Álgebra Lineal

1) Sea el sistema: $\begin{Bmatrix} ay+bx=c \\ cx+az=b \\ bz+cy=a \end{Bmatrix}$ Matrices asociadas: $M^*=\left(\begin{array}{ccc|c} b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{array}\right)$; $|M|=\begin{vmatrix} b & a & 0 \\ c & 0 & a \\ 0 & c & b \end{vmatrix}=-2abc$

Como $a,b,c\neq 0$, $|M|\neq 0 \Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$

Aplicando Cramer: $x=\dfrac{\begin{vmatrix} c & b & a \\ b & 0 & a \\ a & c & b \end{vmatrix}}{-2abc}=\dfrac{a^3-b^2a-c^2a}{-2abc}=\boxed{\dfrac{a^2-b^2-c^2}{-2bc}}$

$$y=\dfrac{\begin{vmatrix} b & c & 0 \\ c & b & a \\ 0 & a & b \end{vmatrix}}{-2abc}=\dfrac{b^3-bc^2-a^2b}{-2abc}=\boxed{\dfrac{b^2-c^2-a^2}{-2ac}}$$

$$z=\dfrac{\begin{vmatrix} b & a & c \\ c & 0 & b \\ 0 & c & a \end{vmatrix}}{-2abc}=\dfrac{c^3-ca^2-b^2c}{-2abc}=\boxed{\dfrac{c^2-a^2-b^2}{-2ab}}$$

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2) Teoría

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3) $|A|=\begin{vmatrix} p & 0 & 0 \\ 1 & p+1 & 1 \\ 1 & 0 & p-1 \end{vmatrix}=p(p+1)(p-1)$

a) $\exists A^{-1} \Leftrightarrow |A|\neq 0 \Leftrightarrow \boxed{p\neq 0,\ p\neq -1,\ p\neq 1}$

b) Si $p=2\Rightarrow A=\begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 1 \\ 1 & 0 & 1 \end{pmatrix}\Rightarrow \boxed{A^{-1}=\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & -1/3 \\ -1/2 & 0 & 1 \end{pmatrix}}$

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4) $\left.\begin{array}{r} 3A+2B=\begin{pmatrix} 8 & 3 \\ 5 & 4 \end{pmatrix} \\ 2A-3B=\begin{pmatrix} 1 & 2 \\ -1 & -6 \end{pmatrix} \end{array}\right\}\Rightarrow \left.\begin{array}{r} 9A+6B=3\cdot\begin{pmatrix} 8 & 3 \\ 5 & 4 \end{pmatrix} \\ 4A-6B=2\cdot\begin{pmatrix} 1 & 2 \\ -1 & -6 \end{pmatrix} \end{array}\right\}$ Sumando ambas ecuaciones:

$$13A=\begin{pmatrix} 24 & 9 \\ 15 & 12 \end{pmatrix}+\begin{pmatrix} 2 & 4 \\ -2 & -12 \end{pmatrix}=\begin{pmatrix} 26 & 13 \\ 13 & 0 \end{pmatrix}\Rightarrow \boxed{A=\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}}$$

De la 1ª ecuación: $3A+2B=\begin{pmatrix}8&3\\5&4\end{pmatrix}\Rightarrow 2B=\begin{pmatrix}8&3\\5&4\end{pmatrix}-3\cdot\begin{pmatrix}2&1\\1&0\end{pmatrix}=\begin{pmatrix}8&3\\5&4\end{pmatrix}-\begin{pmatrix}6&3\\3&0\end{pmatrix}=\begin{pmatrix}2&0\\2&4\end{pmatrix}\Rightarrow$

$$\Rightarrow \boxed{B=\begin{pmatrix}1&0\\1&2\end{pmatrix}}$$

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5) $|A|=\begin{vmatrix}2\cdot2^2&3\cdot2^3&4\cdot2^4&5\cdot2^5\\3\cdot2^3&4\cdot2^4&5\cdot2^5&6\cdot2^6\\4\cdot2^4&5\cdot2^5&6\cdot2^6&7\cdot2^7\\5\cdot2^5&6\cdot2^6&7\cdot2^7&8\cdot2^8\end{vmatrix}=$ Sacando fuera potencias de 2 de las cuatro columnas

$=2^2\cdot2^3\cdot2^4\cdot2^5\begin{vmatrix}2&3&4\cdot&5\\3\cdot2&4\cdot2&5\cdot2&6\cdot2\\4\cdot2^2&5\cdot2^2&6\cdot2^2&7\cdot2^2\\5\cdot2^3&6\cdot2^3&7\cdot2^3&8\cdot2^3\end{vmatrix}=2^{14}\begin{vmatrix}2&3&4\cdot&5\\3\cdot2&4\cdot2&5\cdot2&6\cdot2\\4\cdot2^2&5\cdot2^2&6\cdot2^2&7\cdot2^2\\5\cdot2^3&6\cdot2^3&7\cdot2^3&8\cdot2^3\end{vmatrix}=$ sacando 2 de la 2ª,

2^2 de la 3ª y 2^3 de la 4ª filas $=2^{14}\cdot2\cdot2^2\cdot2^3\begin{vmatrix}2&3&4&5\\3&4&5&6\\4&5&6&7\\5&6&7&8\end{vmatrix}=$ restando a cada fila la anterior:

$$=2^{20}\cdot\begin{vmatrix}2&3&4&5\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{vmatrix}=\boxed{0}$$

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6) $\Delta=\begin{vmatrix}a+b&b+c&c+a\\a'+b'&b'+c'&c'+a'\\a''+b''&b''+c''&c''+a''\end{vmatrix}=\dfrac{C_1-C_2}{C_2-C_3}=\begin{vmatrix}a-c&b-a&c+a\\a'-c'&b'-a'&c'+a'\\a''-c''&b''-a''&c''+a''\end{vmatrix}=C_1+C_3=\begin{vmatrix}2a&b-a&c+a\\2a'&b'-a'&c'+a'\\2a''&b''-a''&c''+a''\end{vmatrix}=$

$$=2\cdot\begin{vmatrix}a&b-a&c+a\\a'&b'-a'&c'+a'\\a''&b''-a''&c''+a''\end{vmatrix}=\frac{C_3-C_1}{C_2+C_1}=2\cdot\begin{vmatrix}a&b&c\\a'&b'&c'\\a''&b''&c''\end{vmatrix}=\boxed{2\Delta_1}$$

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7) $AM=\begin{pmatrix}0&-1\\1&1\end{pmatrix}\cdot\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}-c&-d\\a+c&b+d\end{pmatrix}$; $MA=\begin{pmatrix}a&b\\c&d\end{pmatrix}\cdot\begin{pmatrix}0&-1\\1&1\end{pmatrix}=\begin{pmatrix}b&b-a\\d&d-c\end{pmatrix}$

Si $AM=MA\Rightarrow\begin{pmatrix}-c&-d\\a+c&b+d\end{pmatrix}=\begin{pmatrix}b&b-a\\d&d-c\end{pmatrix}\Rightarrow\left\{\begin{matrix}b=-c\\b-a=-d\\a+c=d\\b+d=d-c\end{matrix}\right\}\Rightarrow\boxed{M=\begin{pmatrix}d-c&-c\\c&d\end{pmatrix}}$

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8) $\begin{cases} 3x-2y+z=m \\ 5x-8y+9z=3 \\ 2x+y-3z=-1 \end{cases} \Rightarrow \; M^*=\left(\begin{array}{ccc|c} 3 & -2 & 1 & m \\ 5 & -8 & 9 & 3 \\ 2 & 1 & -3 & -1 \end{array}\right)$; $|M|=\begin{vmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & -3 \end{vmatrix}=0$ (El sistema no será nunca compatible determinado)

Estudiamos rangos: $rang\left(\begin{array}{ccc|c} 3 & -2 & 1 & m \\ 5 & -8 & 9 & 3 \\ 2 & 1 & -3 & -1 \end{array}\right) = F_1/3 = rang\left(\begin{array}{ccc|c} 1 & -2/3 & 1/3 & m/3 \\ 5 & -8 & 9 & 3 \\ 2 & 1 & -3 & -1 \end{array}\right) = \begin{matrix} F_2-5F_1 \\ F_3-2F_1 \end{matrix} =$

$= rang\left(\begin{array}{ccc|c} 1 & -2/3 & 1/3 & m/3 \\ 0 & -14/3 & 22/3 & 3-(5m/3) \\ 0 & 7/3 & -11/3 & -1-(2m/3) \end{array}\right) = F_2 \leftrightarrow F_3 = rang\left(\begin{array}{ccc|c} 1 & -2/3 & 1/3 & m/3 \\ 0 & 7/3 & -11/3 & -1-(2m/3) \\ 0 & -14/3 & 22/3 & 3-(5m/3) \end{array}\right) = F_3+2F_2 =$

$= rang\left(\begin{array}{ccc|c} 1 & -2/3 & 1/3 & m/3 \\ 0 & 7/3 & -11/3 & -1-(2m/3) \\ 0 & 0 & 0 & 1-3m \end{array}\right)$

<u>Caso 1</u>.- Si $m=\dfrac{1}{3} \Rightarrow rang(M)=2=rang(M^*) \Rightarrow SCI$

<u>Caso 2</u>.- Si $m\neq\dfrac{1}{3} \Rightarrow rang(M)=2\neq rang(M^*)=3 \Rightarrow SI$

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9) Veamos si $A\cdot B$ es una matriz ortogonal. Es decir si $(A\cdot B)\cdot(A\cdot B)^t=I$

Como $(A\cdot B)^t=B^t\cdot A^t$, $(A\cdot B)\cdot(A\cdot B)^t=(A\cdot B)\cdot B^t\cdot A^t=A\cdot B\cdot B^t\cdot A^t=$ como B es ortogonal

$=A\cdot I\cdot A^t=A\cdot A^t=$ como A es ortogonal $=I$

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10) Sea S sistema de **m** ecuaciones con **n** incógnitas, con matrices asociadas: $\left\langle \begin{matrix} A\in M_{m\,n} \\ A^*\in M_{m\,n+1} \end{matrix} \right.$

Como es compatible determinado, $rang(A)=rang(A^*)=n$ (por Rouche) , luego $m\geq n \Rightarrow \left\langle \begin{matrix} m-1=n \\ o \\ m-1>n \end{matrix} \right.$

Consideremos ahora S' con **m-1** ecuaciones y **n** incógnitas, de matrices: $\left\langle \begin{matrix} B\in M_{m-1\,n} \\ B^*\in M_{m-1\,n+1} \end{matrix} \right.$

- <u>Caso 1</u>.- Si $m-1=n$, $B\in M_n$ y $B^*\in M_{n\,n+1}$, como $rang(A)=n \Rightarrow rang(B)=n=rang(B^*)$

  Luego el sistema sería Compatible Determinado
- <u>Caso 2</u>.- Si $m-1>n$, el sistema sería Compatible Indeterminado

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11) Sea el sistema $\begin{cases} x+2y+z=0 \\ y+2z+t=0 \\ 2x+2\lambda y-t=0 \end{cases}$, matriz de coeficientes: $M=\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 2\lambda & 0 & -1 \end{pmatrix}$

Estudiamos rango: $rang(M) = rang\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & 2\lambda & 0 & -1 \end{pmatrix} = F_2 - 2F_1 = rang\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 2\lambda - 4 & -2 & -1 \end{pmatrix} =$

$= F_3 - (2\lambda - 4)F_2 = rang\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 6-4\lambda & 3-2\lambda \end{pmatrix}$

a) Si $rang(M) = 2 \Rightarrow 6 - 4\lambda = 0 \Rightarrow 3 - 2\lambda = 0 \Rightarrow \boxed{\lambda = \frac{3}{2}}$

b) Si $\lambda = 0 \Rightarrow rang(M) = 3 < 4$, sistema compatible indeterminado.

Sustituyendo $\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 6 & 3 \end{pmatrix} \Rightarrow \begin{Bmatrix} x + 2y + z = 0 \Rightarrow x = -z \\ y + 2z + t = 0 \Rightarrow y = 0 \\ 6z + 3t = 0 \Rightarrow t = -2z \end{Bmatrix}$. Sol: $\boxed{\{(-z, 0, z, -2z) | z \in R\}}$

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12) Sean x = Kg de café tipo A, y = Kg de café tipo B y z = Kg de café tipo C

Se obtiene el sistema: $\begin{Bmatrix} x + y + z = 1050 \\ z = 2(x + y) \\ 980x + 875y + 950z = 940 \cdot 1050 \end{Bmatrix} \Rightarrow$ de la 2ª ecuación $x + y = \frac{z}{2}$, sustituyendo

En la 1ª ecuación: $z + \frac{z}{2} = 1050 \Rightarrow \boxed{z = 700}$; de donde $x + y = 350 \Rightarrow y = 350 - x$. Sustituyendo en la 3ª

ecuación $980x + 875(350 - x) + 950 \cdot 700 = 987000 \Rightarrow 105x = 15750 \Rightarrow \boxed{x = 150}$

Por último: $y = 350 - x \Rightarrow \boxed{y = 200}$

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13) Sean x = monedas de 100 de A, y = monedas de 100 de B, y z = monedas de 100 de C

Obtenemos el sistema: $\begin{Bmatrix} x + y + z = 36 \\ x = y + z + 2 \\ x + 1 = 2(y - 1) \end{Bmatrix} \Rightarrow$ Operando: $\begin{Bmatrix} x + y + z = 36 \\ x - y - z = 2 \\ x - 2y = -3 \end{Bmatrix} \Rightarrow$ Matricialmente:

$$M^* = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 1 & -1 & -1 & 2 \\ 1 & -2 & 0 & -3 \end{array}\right) \approx F_i - F_1 \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & -2 & -2 & -34 \\ 0 & -3 & -1 & -39 \end{array}\right) \approx F_2/(-2) \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 1 & 17 \\ 0 & -3 & -1 & -39 \end{array}\right) \approx F_3 + 3F_2 \approx$$

$$\approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 36 \\ 0 & 1 & 1 & 17 \\ 0 & 0 & 2 & 12 \end{array}\right) \Rightarrow \begin{Bmatrix} x + y + z = 36 \\ y + z = 17 \\ 2z = 12 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x + y + z = 36 \\ y + z = 17 \\ z = 6 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x + y = 30 \\ y = 17 - 6 = 11 \end{Bmatrix} \Rightarrow x = 19$$

Luego: $\boxed{x = 19}$; $\boxed{y = 11}$; $\boxed{z = 6}$

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14) a) $rang(A)=rang\begin{pmatrix}1 & 0\\ 1 & -1\\ -2 & 2\end{pmatrix}=2$, pues $\begin{vmatrix}1 & 0\\ 1 & -1\end{vmatrix}=-1\neq 0$;

$rang(B)=rang\begin{pmatrix}-2 & 2 & 0\\ 3 & -1 & 1\end{pmatrix}=2$; pues $\begin{vmatrix}-2 & 2\\ 3 & -1\end{vmatrix}=-4\neq 0$

$$A\cdot B=\begin{pmatrix}1 & 0\\ 1 & -1\\ -2 & 2\end{pmatrix}\cdot\begin{pmatrix}-2 & 2 & 0\\ 3 & -1 & 1\end{pmatrix}=\begin{pmatrix}-2 & 2 & 0\\ -5 & 3 & -1\\ 0 & 0 & 0\end{pmatrix}\Rightarrow rang(AB)=2\neq rang(A)\cdot rang(B)=4$$

b) $X\cdot A=I\Rightarrow\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ 1 & -1\\ -2 & 2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Rightarrow X=\begin{pmatrix}a+b-2c & -b+2c\\ d+e-2f & -e+2f\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Rightarrow$

$\Rightarrow\left\{\begin{array}{c}a+b-2c=1\\ -b+2c=0\Rightarrow b=2c\\ d+e-2f=0\\ -e+2f=1\Rightarrow e=2f-1\end{array}\right\}\Rightarrow$ sustituyendo: $\left\{\begin{array}{c}a+2c-2c=1\\ d+2f-1-2f=0\end{array}\right\}\Rightarrow a=d=1$

$$\boxed{X=\begin{pmatrix}1 & 2c & c\\ 1 & 2f-1 & f\end{pmatrix}}$$

c) $A\cdot Y=B^t\Rightarrow\begin{pmatrix}1 & 0\\ 1 & -1\\ -2 & 2\end{pmatrix}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}-2 & 3\\ 2 & -1\\ 0 & 1\end{pmatrix}\Rightarrow\begin{pmatrix}a & b\\ a-c & b-d\\ 2a+2c & 2b+2d\end{pmatrix}=\begin{pmatrix}-2 & 3\\ 2 & -1\\ 0 & 1\end{pmatrix}\Rightarrow$

$\Rightarrow\left\{\begin{array}{c}a=-2\\ b=3\\ a-c=2\\ b-d=-1\\ 2a+2c=0\\ 2b+2d=1\end{array}\right\}$ sustituyendo $\left\{\begin{array}{c}a=-2\\ b=3\\ c=-2\\ d=4\\ 2a+2c=-8\neq 0\\ 2b+2d=10\neq 0\end{array}\right\}\Rightarrow$ No existe dicha matriz Y

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15) a) $AB=\begin{pmatrix}1 & 2 & \lambda\\ 1 & -1 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & 3\\ \lambda & 0\\ 0 & 2\end{pmatrix}=\begin{pmatrix}1+2\lambda & 2\lambda+3\\ 1-\lambda & 1\end{pmatrix}\Rightarrow|AB|=-(2\lambda+2)(1-\lambda)=0\Rightarrow\left\langle\begin{array}{l}\lambda=-1\\ \lambda=1\end{array}\right.$

La matriz AB es invertible si $|AB|\neq 0\Leftrightarrow\boxed{\lambda\neq -1}$ y $\boxed{\lambda\neq 1}$

b) $BA=\begin{pmatrix}1 & 3\\ \lambda & 0\\ 0 & 2\end{pmatrix}\cdot\begin{pmatrix}1 & 2 & \lambda\\ 1 & -1 & -1\end{pmatrix}=\begin{pmatrix}4 & -1 & \lambda-3\\ \lambda & 2\lambda & \lambda^2\\ 2 & -2 & -2\end{pmatrix}\Rightarrow|BA|=0$, luego **nunca** es invertible

c) $A\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}\Rightarrow\begin{pmatrix}1&2&\lambda\\1&-1&-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}\Rightarrow\begin{pmatrix}x+2y+\lambda z\\x-y-z\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}\Rightarrow\left\{\begin{matrix}x+2y+\lambda z=a\\x-y-z=b\end{matrix}\right\}$, sistema que no puede ser nunca compatible determinado, pues tiene 3 incógnitas y solo 2 ecuaciones

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16) $\left\{\begin{matrix}ax+2y+6z=0\\2x+ay+4z=2\\2x+ay+6z=a-2\end{matrix}\right.\Rightarrow$ Matrices asociadas: $M^*=\left(\begin{array}{ccc|c}a&2&6&0\\2&a&4&2\\2&a&6&a-2\end{array}\right)$, $|M|=\begin{vmatrix}a&2&6\\2&a&4\\2&a&6\end{vmatrix}=2a^2-8$

$|M|=0\Rightarrow 2a^2-8=0\Rightarrow a^2=4\Rightarrow\left\langle\begin{matrix}a=-2\\a=2\end{matrix}\right.$

a) <u>Caso 1</u>.- Si $a\neq\pm2$, $|M|=0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$

<u>Caso 2</u>.- Si $a=-2$, $rang\left(\begin{array}{ccc|c}-2&2&6&0\\2&-2&4&2\\2&-2&6&-4\end{array}\right)=F_i+F_1=rang\left(\begin{array}{ccc|c}-2&2&6&0\\0&0&10&2\\0&0&12&-4\end{array}\right)=\begin{matrix}F_2/10\\F_3/12\end{matrix}=$

$=rang\left(\begin{array}{ccc|c}-2&2&6&0\\0&0&1&1/5\\0&0&1&-1/3\end{array}\right)=F_3-F_2=rang\left(\begin{array}{ccc|c}-2&2&6&0\\0&0&1&1/5\\0&0&0&-8/15\end{array}\right)\Rightarrow rang(M)=2\neq rang(M^*)=3\Rightarrow SI$

<u>Caso 3</u> .- Si $a=2$, $rang\left(\begin{array}{ccc|c}2&2&6&0\\2&2&4&2\\2&2&6&0\end{array}\right)=F_i-F_1=rang\left(\begin{array}{ccc|c}2&2&6&0\\0&0&-2&2\\0&0&0&0\end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$

b) Solución si $a=2$, $M^*\approx\left(\begin{array}{ccc|c}2&2&6&0\\0&0&-2&2\\0&0&0&0\end{array}\right)\Rightarrow\left\{\begin{matrix}2x+2y+6z=0\\-2z=2\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}x+y+3z=0\\z=-1\end{matrix}\right\}$.

Si $y=\lambda\Rightarrow x=-\lambda-3z=3-\lambda\Rightarrow$ Solución: $\boxed{\{(3-\lambda,\lambda,-1)|\lambda\in R\}}$

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17) $\Delta=\begin{vmatrix}a&a&a&a\\2&a&a&a\\3&2&a&a\\4&3&2&a\end{vmatrix}=\begin{matrix}F_1-F_2\\F_2-F_3\\F_3-F_4\end{matrix}=\begin{vmatrix}a-2&0&0&0\\-1&a-2&0&0\\-1&-1&a-2&0\\4&3&2&a\end{vmatrix}=\boxed{a(a-2)^3}$

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18) Sean x= billetes de 1000 , y= billetes de 2000, y z= billetes de 5000

Obtenemos el sistema: $\left\{\begin{matrix}x+y+z=95\\x+2y+5z=200\\x=2y\end{matrix}\right\}\Rightarrow$ Sustituyendo: $\left\{\begin{matrix}2y+y+z=95\\2y+2y+5z=200\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}3y+z=95\\5y+5z=200\end{matrix}\right\}$

Resolviendo: $\boxed{x=50}$; $\boxed{y=25}$; $\boxed{z=20}$

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19) a) $\begin{cases} (a+1)x+2y+z=a+3 \\ ax+y=a \\ ax+3y+z=a+2 \end{cases} \Rightarrow$ Matrices: $M^*=\left(\begin{array}{ccc|c} a+1 & 2 & 1 & a+3 \\ a & 1 & 0 & a \\ a & 3 & 1 & a+2 \end{array}\right)$, $|M|=\begin{vmatrix} a+1 & 2 & 1 \\ a & 1 & 0 \\ a & 3 & 1 \end{vmatrix}=a+1=0 \Rightarrow a=-1$

- <u>Caso 1</u>.- Si $a \neq -1$, $|M| \neq 0 \Rightarrow rang(M)=3=rang(M^*) \Rightarrow SCD$
- <u>Caso 2</u>.- Si $a=-1$, $rang\left(\begin{array}{ccc|c} 0 & 2 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ -1 & 3 & 1 & 1 \end{array}\right)=F_2 \leftrightarrow F_1 = rang\left(\begin{array}{ccc|c} -1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ -1 & 3 & 1 & 1 \end{array}\right)=F_3-F_1=$

$$=rang\left(\begin{array}{ccc|c} -1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right)=F_3-F_2=rang\left(\begin{array}{ccc|c} -1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow rang(M)=2=rang(M^*) \Rightarrow SCI$$

b) El sistema es compatible determinado si $a \neq -1$. Resolvemos por Cramer:

- $x=\dfrac{\begin{vmatrix} a+3 & 2 & 1 \\ a & 1 & 0 \\ a+2 & 3 & 1 \end{vmatrix}}{a+1}=\dfrac{a+1}{a+1} \Rightarrow \boxed{x=1}$
- $y=\dfrac{\begin{vmatrix} a+1 & a+3 & 1 \\ a & a & 0 \\ a & a+2 & 1 \end{vmatrix}}{a+1}=\dfrac{0}{a+1} \Rightarrow \boxed{y=0}$
- $z=\dfrac{\begin{vmatrix} a+1 & 2 & a+3 \\ a & 1 & a \\ a & 3 & a+2 \end{vmatrix}}{a+1}=\dfrac{2a+2}{a+1} \Rightarrow \boxed{z=2}$

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20) Sean las matrices $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ y $B=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, es claro que $Traza(A)=a+d$; $Traza(B)=e+h$

a) $Traza(A+B)=Traza\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]=Traza\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}=a+e+d+h=Traza(A)+Traza(B)$

b) $Traza(A\cdot B)=Traza\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]=Traza\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}=ae+bg+cf+dh$

$Traza(B\cdot A)=Traza\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]=Traza\begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}=ea+cf+gb+hd$ (Iguales)

c) Si $AB-BA=I \Rightarrow Traza(AB-BA)=Traza(I)=2$; por a) $Traza(AB-BA)=Traza(AB)-Traza(BA)$, y por apartado b) $Traza(AB)-Traza(BA)=0$, con lo que $2=0$

d) Sean las matrices $A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ y $B=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A\cdot B=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$Traza(A\cdot B)=0$, $Traza(A)=1$ y $Traza(B)=1$, luego $Traza(A)\cdot Traza(B)=1\cdot 1=1\neq 0$

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21) a) $\begin{cases} ax+y+z=(a-1)(a+2)\\ x+ay+z=(a-1)^2(a+2)\\ x+y+az=(a-1)^3(a+2)\end{cases}\Rightarrow$ Matrices: $M^*=\left(\begin{array}{ccc|c} a & 1 & 1 & (a-1)(a+2)\\ 1 & a & 1 & (a-1)^2(a+2)\\ 1 & 1 & a & (a-1)^3(a+2)\end{array}\right)$

$$|M|=\begin{vmatrix} a & 1 & 1\\ 1 & a & 1\\ 1 & 1 & 1\end{vmatrix}=(a-1)^2(a+2)=0$$

- Caso 1.- Si $a\neq 1, a\neq -2$, $|M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- Caso 2.- Si $a=1$, el sistema es homogéneo, luego compatible
- Caso 3.- Si $a=-2$, el sistema es homogéneo, luego compatible

c) Si $a=-2$, $rang(M^*)=rang\begin{pmatrix} -2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{pmatrix}=F_2\leftrightarrow F_1=rang\begin{pmatrix} 1 & -2 & 1\\ -2 & 1 & 1\\ 1 & 1 & -2\end{pmatrix}=\begin{matrix}F_2+2F_1\\ F_3-F_1\end{matrix}=$

$=rang\begin{pmatrix} 1 & -2 & 1\\ 0 & -3 & 3\\ 0 & 3 & -3\end{pmatrix}=F_3+F_2=rang\begin{pmatrix} 1 & -2 & 1\\ 0 & -3 & 3\\ 0 & 0 & 0\end{pmatrix}\Rightarrow\left\{\begin{matrix} x-2y+z=0\\ -3y+3z=0\end{matrix}\right\}\Rightarrow\left\{\begin{matrix} x=z\\ y=z\end{matrix}\right\}$, si $z=\lambda$

Solución: $\boxed{\{(\lambda,\lambda,\lambda)|\lambda\in R\}}$

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22) a) $\begin{cases} y+z=1\\ (\lambda-1)x+y+z=\lambda\\ x+(\lambda-1)y-z=0\end{cases}\Rightarrow$ Matrices asociadas: $M=\begin{pmatrix} 0 & 1 & 1\\ \lambda-1 & 1 & 1\\ 1 & \lambda-1 & -1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1\\ \lambda-1 & 1 & 1 & \lambda\\ 1 & \lambda-1 & -1 & 0\end{array}\right)$

$$|M|=\begin{vmatrix} 0 & 1 & 1\\ \lambda-1 & 1 & 1\\ 1 & \lambda-1 & -1\end{vmatrix}=\lambda(\lambda-1)=0\Rightarrow\begin{cases}\lambda=0\\ \lambda=1\end{cases}$$

- Caso 1.- Si $\lambda\neq 0,\lambda\neq 1$, $|M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- Caso 2.- Si $\lambda=0$, $rang\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1\\ -1 & 1 & 1 & 0\\ 1 & -1 & -1 & 0\end{array}\right)=F_3+F_2=rang\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1\\ -1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\Rightarrow rang(M)=2=$

  $=rang(M^*)\Rightarrow SCI$
- Caso 3.- Si $\lambda=1$, $rang\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\\ 1 & 0 & -1 & 0\end{array}\right)=F_1\leftrightarrow F_3=rang\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0\\ 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1\end{array}\right)=F_3-F_2=$

$$rang\begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \Rightarrow rang(M)=2=rang(M^*) \Rightarrow SCI$$

b) Si $\lambda = 0$, $M^* \approx \left(\begin{array}{ccc|c}0 & 1 & 1 & 1\\ -1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{array}\right) \Rightarrow \left\{\begin{array}{c} y+z=1\\ -x+y+z=0\end{array}\right\} \Rightarrow \left\{\begin{array}{c} x=1\\ y=1-z\end{array}\right\}$. Si $z=\lambda$, Sol: $\boxed{\{(1,1-\lambda,\lambda) | \lambda \in R\}}$

c) Si $\lambda = 3$, el sistema es compatible determinado. $M^* \approx \left(\begin{array}{ccc|c}0 & 1 & 1 & 1\\ 2 & 1 & 1 & 3\\ 1 & 2 & -1 & 0\end{array}\right) \approx F_3 \leftrightarrow F_1 \approx \left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 2 & 1 & 1 & 3\\ 0 & 1 & 1 & 1\end{array}\right) \approx$

$$\approx F_2 - 2F_1 \approx \left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 0 & -3 & 3 & 3\\ 0 & 1 & 1 & 1\end{array}\right) \approx F_3/3 \approx \left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 0 & -1 & 1 & 1\\ 0 & 1 & 1 & 1\end{array}\right) \approx F_3 + F_2 \approx \left(\begin{array}{ccc|c}1 & 2 & -1 & 0\\ 0 & -1 & 1 & 1\\ 0 & 0 & 2 & 2\end{array}\right) \Rightarrow z=1$$

Sustituyendo: $\left\{\begin{array}{c} x+2y-1=0 \Rightarrow x=1\\ -y+1=1 \Rightarrow y=0\end{array}\right\}$. Sol: $\boxed{x=1}$, $\boxed{y=0}$, $\boxed{z=1}$

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23) a) $\left\{\begin{array}{l} x+y+5z=0\\ 2x-kz=0\\ x-y+z=0\end{array}\right. \Rightarrow$ Matriz asociada: $M=\begin{pmatrix}1 & 1 & 5\\ 2 & 0 & -k\\ 1 & -1 & 1\end{pmatrix} \Rightarrow |M| = \begin{vmatrix}1 & 1 & 5\\ 2 & 0 & -k\\ 1 & -1 & 1\end{vmatrix} = -2k-12=0 \Rightarrow k=-6$

- Caso 1.- Si $k \neq -6$, $|M| \neq 0 \Rightarrow rang(M)=3=rang(M^*) \Rightarrow SCD$. Solución: $\boxed{x=y=z=0}$
- Caso 2.- Si $k=-6$, $rang\begin{pmatrix}1 & 1 & 5\\ 2 & 0 & 6\\ 1 & -1 & 1\end{pmatrix} = \begin{array}{c}F_2-2F_1\\ F_3-F_1\end{array} = rang\begin{pmatrix}1 & 1 & 5\\ 0 & -2 & -4\\ 0 & -2 & -4\end{pmatrix} = F_3 - F_2 =$

$$= rang\begin{pmatrix}1 & 1 & 5\\ 0 & -2 & -4\\ 0 & 0 & 0\end{pmatrix} \Rightarrow rang(M)=2=rang(M^*) \Rightarrow SCI$$

Como $M^* \approx \begin{pmatrix}1 & 1 & 5\\ 0 & -2 & -4\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \left\{\begin{array}{c} x+y+5z=0 \Rightarrow x=-3z\\ -2y-4z=0 \Rightarrow y=-2z\end{array}\right\} \Rightarrow z=\lambda$, Sol: $\boxed{\{(-3\lambda,-2\lambda,\lambda) | \lambda \in R\}}$

b) $\left\{\begin{array}{c} x+y+5z=0\\ 2x-3z=0\\ x-y+z=0\\ x+2y+2\lambda z=\lambda\end{array}\right.$ Matrices asociadas: $M=\begin{pmatrix}1 & 1 & 5\\ 2 & 0 & -3\\ 1 & -1 & 1\\ 1 & 2 & 2\lambda\end{pmatrix}$ y $M^* = \begin{pmatrix}1 & 1 & 5 & 0\\ 2 & 0 & -3 & 0\\ 1 & -1 & 1 & 0\\ 1 & 2 & 2\lambda & \lambda\end{pmatrix}$

$$|M^*| = \begin{vmatrix}1 & 1 & 5 & 0\\ 2 & 0 & -3 & 0\\ 1 & -1 & 1 & 0\\ 1 & 2 & 2\lambda & \lambda\end{vmatrix} = \text{desarrollando por 4ª columna } \lambda \cdot \begin{vmatrix}1 & 1 & 5\\ 2 & 0 & -3\\ 1 & -1 & 1\end{vmatrix} = \lambda(-2\cdot 3 - 12) = -18\lambda$$

(Ver determinante de la matriz M anterior)

- Caso 1.- Si $\lambda \neq 0$, $|M^*| \neq 0 \Rightarrow rang(M^*) = 4 > rang(M) \Rightarrow SI$.

- Caso 2.- Si $\lambda = 0$, $rang(M^*) = \begin{pmatrix} 1 & 1 & 5 & 0 \\ 2 & 0 & -3 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix} = rang \begin{pmatrix} 1 & 1 & 5 \\ 2 & 0 & -3 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix} = \begin{matrix} F_2 - 2F_1 \\ F_3 - F_1 \\ F_4 - F_1 \end{matrix} =$

$$= rang \begin{pmatrix} 1 & 1 & 5 \\ 0 & 2 & -13 \\ 0 & -2 & -4 \\ 0 & 1 & -5 \end{pmatrix} = F_2 \leftrightarrow F_4 = rang \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -5 \\ 0 & -2 & -4 \\ 0 & 2 & -13 \end{pmatrix} = \begin{matrix} F_3 + 2F_2 \\ F_4 - 2F_2 \end{matrix} = rang \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{pmatrix} =$$

$$= rang \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 6 \end{pmatrix} \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$$, Sol: $\boxed{x = y = z = 0}$

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24) a) $\begin{cases} x + y + 2z = 2 \\ 2x - y + 3z = 2 \\ 5x - y + az = 6 \end{cases} \Rightarrow$ Matrices asociadas: $M = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & a \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & 6 \end{array}\right)$

$$|M| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 3 \\ 5 & -1 & a \end{vmatrix} = 24 - 3a = 0 \Rightarrow a = 8$$

- Caso 1.- Si $a \neq 8$, $|M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$.

- Caso 2.- Si $a = 8$, $rang(M^*) = rang \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & 8 & 6 \end{array}\right) = \begin{matrix} F_2 - 2F_1 \\ F_3 - 5F_1 \end{matrix} = rang \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & -2 & -4 \end{array}\right) =$

$$= F_3 - 2F_2 = rang \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow rang(M) = 2 = rang(*) \Rightarrow SCI$$

b) Si $a = 8$, $M^* \approx \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left\{ \begin{array}{l} x + y + 2z = 2 \Rightarrow x = 5y - 2 \\ -3y - z = -2 \Rightarrow z = 2 - 3y \end{array} \right\}$ Si $y = \lambda$,

Solución : $\boxed{\{(5\lambda - 2, \lambda, 2 - 3\lambda) | \lambda \in R\}}$

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25) a) Sea $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow A^3 = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \boxed{A^k = \begin{pmatrix} 1 & k & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

$$A^k X = BC \Rightarrow A^k X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix} \Rightarrow (A^k)^{-1} A^k X = (A^k)^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix} \Rightarrow X = (A^k)^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix}$$

Es decir, $X = A^{-k} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -k & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix} \Rightarrow \boxed{X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{pmatrix}}$

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26) a) $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{pmatrix}$; $|M^*| = \begin{vmatrix} 1 & 1 & 1 & \lambda \\ 1 & 1 & \lambda & 1 \\ 1 & \lambda & 1 & 1 \\ \lambda & 1 & 1 & 1 \end{vmatrix} = (3+\lambda)(1-\lambda)^3 = 0 \Rightarrow \left\langle \begin{matrix} \lambda = 1 \\ \lambda = -3 \end{matrix} \right.$

- <u>Caso 1</u>.- Si $\lambda \neq 1, \lambda = -3$, $|M^*| \neq 0 \Rightarrow rang(M^*) = 4 > rang(M) \Rightarrow SI$.
- <u>Caso 2</u>.- Si $\lambda = 1$, $rang(M^*) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) = F_i - F_1 = rang(M^*) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$

  $rang(M^*) = 1 = rang(M) \Rightarrow SCI$
- <u>Caso 3</u>.- Si $\lambda = -3$, $rang(M^*) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \\ -3 & 1 & 1 & 1 \end{array}\right) = \begin{matrix} F_2 - F_1 \\ F_3 - F_1 \\ F_4 + 3F_1 \end{matrix} = rang \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 0 & -4 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 4 & 4 & -8 \end{array}\right) \Rightarrow$

$$= F_4 + F_3 = rang \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 0 & -4 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 4 & -4 \end{array}\right) \Rightarrow F_4 + F_2 = rang \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 0 & -4 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$$

b) Si $\lambda = -3$, sabemos que el sistema es compatible determinado

$M^* \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 0 & -4 & 4 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 0 & -4 & 4 \\ 0 & -4 & 0 & 4 \end{array}\right) \Rightarrow \left\{ \begin{array}{c} x + y + z = -3 \\ -4z = 4 \Rightarrow z = -1 \\ -4y = 4 \Rightarrow y = -1 \end{array} \right\} \Rightarrow$ Sol: $\boxed{x = y = z = -1}$

c) Si $\lambda = 3 \Rightarrow SCI \Rightarrow M^* \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \{x + y + z = 1\} \Rightarrow$ Si $\left\{ \begin{array}{c} y = s \\ z = t \end{array} \right\} \Rightarrow$ Sol : $\boxed{\{(1 - s - t, s, t) | t, s \in R\}}$

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27) a) $\left\{ \begin{array}{c} ax + y + 4z = 1 \\ -x + ay - 2z = 1 \\ y + z = a \end{array} \right. \Rightarrow$ Matrices asociadas: $M = \begin{pmatrix} a & 1 & 4 \\ -1 & a & -2 \\ 0 & 1 & 1 \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} a & 1 & 4 & 1 \\ -1 & a & -2 & 1 \\ 0 & 1 & 1 & a \end{array}\right)$

$$|M|=\begin{vmatrix} a & 1 & 4 \\ -1 & a & -2 \\ 0 & 1 & 1 \end{vmatrix}=a^2+2a-3=0 \Rightarrow \left\langle \begin{array}{l} a=1 \\ a=-3 \end{array} \right.$$

- Caso 1.- Si $a\neq 1, a\neq -3$, $|M|\neq 0 \Rightarrow rang(M)=3=rang(M^*) \Rightarrow SCD$.
- Caso 2.- Si $a=1$, $rang(M^*)=rang\begin{pmatrix} 1 & 1 & 4 & | & 1 \\ -1 & 1 & -2 & | & 1 \\ 0 & 1 & 1 & | & 1 \end{pmatrix}=F_2+F_1=rang\begin{pmatrix} 1 & 1 & 4 & | & 1 \\ 0 & 2 & 2 & | & 2 \\ 0 & 1 & 1 & | & 1 \end{pmatrix}=F_2/2=$

$$=rang\begin{pmatrix} 1 & 1 & 4 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 1 & 1 & | & 1 \end{pmatrix}=F_3-F_2=rang\begin{pmatrix} 1 & 1 & 4 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

- Caso 3.- Si $a=-3$, $rang\begin{pmatrix} -3 & 1 & 4 & | & 1 \\ -1 & -3 & -2 & | & 1 \\ 0 & 1 & 1 & | & -3 \end{pmatrix}=F_2\leftrightarrow F_1=rang\begin{pmatrix} -1 & -3 & -2 & | & 1 \\ -3 & 1 & 4 & | & 1 \\ 0 & 1 & 1 & | & -3 \end{pmatrix}=F_2-3F_1=$

$$=rang\begin{pmatrix} -1 & -3 & -2 & | & 1 \\ 0 & 10 & 10 & | & -2 \\ 0 & 1 & 1 & | & -3 \end{pmatrix}=F_2/10=rang\begin{pmatrix} -1 & -3 & -2 & | & 1 \\ 0 & 1 & 1 & | & -1/5 \\ 0 & 1 & 1 & | & -3 \end{pmatrix}=F_3-F_2=rang\begin{pmatrix} -1 & -3 & -2 & | & 1 \\ 0 & 1 & 1 & | & -1/5 \\ 0 & 0 & 0 & | & -14/5 \end{pmatrix}$$

$$\Rightarrow rang(M)=2\neq 3=rang(M^*)\Rightarrow SI$$

b) Si $a=1$, $M^*\approx\begin{pmatrix} 1 & 1 & 4 & | & 1 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}\Rightarrow\Rightarrow\left\{\begin{array}{c} x+y+4z=1 \\ y+z=1\Rightarrow \end{array}\right\}\Rightarrow$ Si $z=\lambda\Rightarrow\left\{\begin{array}{c} x=-3\lambda \\ y=1-\lambda \end{array}\right\}\Rightarrow$ Sol: $\boxed{\{(-3\lambda,-1-\lambda,\lambda) | \lambda\in R\}}$

c) Si $a=2$, el sistema es compatible determinado.

$rang(M^*)=rang\begin{pmatrix} 2 & 1 & 4 & | & 1 \\ -1 & 2 & -2 & | & 1 \\ 0 & 1 & 1 & | & 2 \end{pmatrix}=F_1+2F_2=rang\begin{pmatrix} 0 & 5 & 0 & | & 3 \\ -1 & 2 & -2 & | & 1 \\ 0 & 1 & 1 & | & 2 \end{pmatrix}\Rightarrow\left\{\begin{array}{c} y=3/5 \\ -x+2y-2z=1 \\ y+z=2 \end{array}\right\}$ Sust. el valor de y

$\Rightarrow\left\{\begin{array}{c} y=3/5 \\ -x+6/5-2z=1 \\ z=7/5 \end{array}\right\}\Rightarrow$ Solución: $\boxed{x=-\frac{13}{5}}$, $\boxed{y=\frac{3}{5}}$, $\boxed{z=\frac{7}{5}}$

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28) a) Sea $A=\begin{pmatrix} 0 & 3 & 4 \\ 1 & -4 & -5 \\ -1 & 3 & 4 \end{pmatrix}\Rightarrow A^2=\begin{pmatrix} -1 & 0 & 1 \\ 1 & 4 & 4 \\ -1 & -3 & -3 \end{pmatrix}\Rightarrow A^3=\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}=-I_3$

b) Como $A^3=-I\Rightarrow AA^2=-I\Rightarrow -AA^2=I\Rightarrow A(-A^2)=I\Rightarrow$ multiplicando por A^{-1} a los dos lados:

$\Rightarrow A^{-1}A(-A^2)=A^{-1}I\Rightarrow \boxed{A^{-1}=-A^2}$

c) $A^{100}=A^{333+1}=A^{333}A=(A^3)^{33}A=$ por apartado a) $=(-I)^{33}A=-IA=-A\Rightarrow \boxed{A^{100}=-A}$

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29) Sean $x =$ edad madre ; $y =$ edad primer hijo ; $z =$ edad segundo hijo. Se plantea el sistema:

$$\left\{\begin{matrix} x-14=5(y+z-14-14) \\ x+10=y+10+z+10 \\ z+x-y=42 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x-5y-5z=-126 \\ x-y-z=10 \\ x-y+z=42 \end{matrix}\right\} \Rightarrow \text{ Matrices asociadas } M^*=\left(\begin{array}{ccc|c} 1 & -5 & -5 & -126 \\ 1 & -1 & -1 & 10 \\ 1 & -1 & 1 & 42 \end{array}\right)$$

Diagonalizando: $$M^*=\left(\begin{array}{ccc|c} 1 & -5 & -5 & -126 \\ 1 & -1 & -1 & 10 \\ 1 & -1 & 1 & 42 \end{array}\right) \approx F_i - F_1 \approx \left(\begin{array}{ccc|c} 1 & -5 & -5 & -126 \\ 0 & 4 & 4 & 136 \\ 0 & 4 & 6 & 168 \end{array}\right) \approx F_3 - F_2 \approx \left(\begin{array}{ccc|c} 1 & -5 & -5 & -126 \\ 0 & 4 & 4 & 136 \\ 0 & 0 & 2 & 32 \end{array}\right) \Rightarrow$$

$$\Rightarrow \left\{\begin{matrix} x-5y-5z=-126 \Rightarrow x=44 \\ 4y+4z=136 \Rightarrow y=18 \\ z=16 \end{matrix}\right\} \Rightarrow \text{ Solución: } \boxed{x=44}\,,\ \boxed{y=18}\,,\ \boxed{z=16}$$

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30) $$rang(A)=rang\begin{pmatrix} 2 & 0 & a & 2 \\ -1 & 0 & -1 & 3 \\ 5 & a+4 & -4 & -3 \end{pmatrix} = F_1 \leftrightarrow F_2 = rang\begin{pmatrix} -1 & 0 & -1 & 3 \\ 2 & 0 & a & 2 \\ 5 & a+4 & -4 & -3 \end{pmatrix} = \begin{matrix} F_2+2F_1 \\ F_3+5F_1 \end{matrix} =$$

$$= rang\begin{pmatrix} -1 & 0 & -1 & 3 \\ 0 & 0 & a-2 & 8 \\ 0 & a+4 & -9 & 12 \end{pmatrix} = F_3 \leftrightarrow F_2 = rang\begin{pmatrix} -1 & 0 & -1 & 3 \\ 0 & a+4 & -9 & 12 \\ 0 & 0 & a-2 & 8 \end{pmatrix}$$

- Caso 1.- Si $\boxed{a \neq -4,\ rang(A)=3}$
- Caso 2.- Si $a=-4$, $rang(A)=rang\begin{pmatrix} -1 & 0 & -1 & 3 \\ 0 & 0 & -9 & 12 \\ 0 & 0 & -6 & 8 \end{pmatrix} = \begin{matrix} F_2/3 \\ F_3/2 \end{matrix} = rang\begin{pmatrix} -1 & 0 & -1 & 3 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -3 & 4 \end{pmatrix} = F_3 - F_2 =$

$$= rang\begin{pmatrix} -1 & 0 & -1 & 3 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \boxed{a=-4,\ rang(A)=2}$$

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31) a) $$\left\{\begin{matrix} x & -y & & = & 2 \\ ax & +y & +2z & = & 0 \\ x & -y & +az & = & 1 \end{matrix}\right. \Rightarrow \text{ Matrices asociadas: } M=\begin{pmatrix} 1 & -1 & 0 \\ a & 1 & 2 \\ 1 & -1 & a \end{pmatrix} \text{ y } M^*=\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ a & 1 & 2 & 0 \\ 1 & -1 & a & 1 \end{array}\right)$$

$$|M|=\begin{vmatrix} 1 & -1 & 0 \\ a & 1 & 2 \\ 1 & -1 & a \end{vmatrix} = a(a+1)=0 \Rightarrow \left\langle \begin{matrix} a=-1 \\ a=0 \end{matrix} \right.$$

- Caso 1.- Si $a \neq -1, a \neq 0$, $|M| \neq 0 \Rightarrow rang(M)=3=rang(M^*) \Rightarrow SCD$.
- Caso 2.- Si $a=0$, $rang(M^*)=rang\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) = F_3 - F_1 = rang\left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right) \Rightarrow$

$$\Rightarrow rang(M)=2 \neq 3 = rang(M^*) \Rightarrow SI$$

- Caso 3.- Si $a=-1$, $rang(M^*)=rang\left(\begin{array}{ccc|c}1&-1&0&2\\-1&1&2&0\\1&-1&-1&1\end{array}\right)=\begin{matrix}F_2+F_1\\F_3-F_1\end{matrix}=rang\left(\begin{array}{ccc|c}1&-1&0&2\\0&0&2&2\\0&0&-1&-1\end{array}\right)=F_2\leftrightarrow F_3=$

$$=rang\left(\begin{array}{ccc|c}1&-1&0&2\\0&0&-1&-1\\0&0&2&2\end{array}\right)=F_3+2F_2=rang\left(\begin{array}{ccc|c}1&-1&0&2\\0&0&-1&-1\\0&0&0&0\end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

b) Si $a=-1$, $M^*\approx\left(\begin{array}{ccc|c}1&-1&0&2\\0&0&-1&-1\\0&0&0&0\end{array}\right)\Rightarrow\left\{\begin{array}{c}x-y=2\\-z=-1\Rightarrow z=1\end{array}\right\}\Rightarrow$ Si $y=\lambda$, Sol: $\boxed{\{(2+\lambda,\lambda,1)|\lambda\in R\}}$

c) Si $a=2$, el sistema es compatible determinado.

$$M^*\approx\left(\begin{array}{ccc|c}1&-1&0&2\\2&1&2&0\\1&-1&2&1\end{array}\right)=\begin{matrix}F_2-2F_1\\F_3-F_1\end{matrix}\approx\left(\begin{array}{ccc|c}1&-1&0&2\\0&3&2&-4\\0&0&2&-1\end{array}\right)\Rightarrow\left\{\begin{array}{c}x-y=2\\3y+2z=-4\\2z=-1\Rightarrow z=-1/2\end{array}\right\}\Rightarrow \text{Sustituyendo z,}$$

$\Rightarrow\left\{\begin{array}{c}x-y=2\\3y-1=-4\Rightarrow y=-1\end{array}\right\}\Rightarrow$ Solución: $\boxed{x=1}$, $\boxed{y=-1}$, $\boxed{z=-\frac{1}{2}}$

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32) a) $\left\{\begin{array}{rrrcl}x&+y&+\lambda z&=&\lambda^2\\&+y&-z&=&\lambda\\x&+\lambda y&+z&=&\lambda\end{array}\right.\Rightarrow$ Matrices asociadas: $M=\begin{pmatrix}1&1&\lambda\\0&1&-1\\1&\lambda&1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c}1&1&\lambda&\lambda^2\\0&1&-1&\lambda\\1&\lambda&1&\lambda\end{array}\right)$

$|M|=\begin{vmatrix}1&1&\lambda\\0&1&-1\\1&\lambda&1\end{vmatrix}=0$, $\forall\lambda$. Como $\begin{vmatrix}1&1\\0&1\end{vmatrix}=1\neq0\Rightarrow rang(M)=2\ \ \forall\lambda$

$$rang(M^*)=rang\left(\begin{array}{ccc|c}1&1&\lambda&\lambda^2\\0&1&-1&\lambda\\1&\lambda&1&\lambda\end{array}\right)=F_3-F_1=rang\left(\begin{array}{ccc|c}1&1&\lambda&\lambda^2\\0&1&-1&\lambda\\0&\lambda-1&1-\lambda&\lambda-\lambda^2\end{array}\right)=F_3-(\lambda-1)F_2=$$

$$=rang\left(\begin{array}{ccc|c}1&1&\lambda&\lambda^2\\0&1&-1&\lambda\\0&0&0&2\lambda-2\lambda^2\end{array}\right).\ 2\lambda^2-2\lambda=0\Rightarrow2\lambda(\lambda-1)=0\Rightarrow\left\langle\begin{array}{l}\lambda=0\\\lambda=1\end{array}\right.$$

- Caso 1.- Si $\lambda=0,\ \lambda=1$, $\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$.
- Caso 2.- Si $\lambda\neq0,\ \lambda\neq1$, $\Rightarrow rang(M)=2\neq3=rang(M^*)\Rightarrow SI$.

b) Si $\lambda=0$, $M^*\approx\left(\begin{array}{ccc|c}1&1&0&0\\0&1&-1&0\\0&0&0&0\end{array}\right)\Rightarrow\left\{\begin{array}{c}x+y=0\\y-z=0\end{array}\right\}$ Si $z=\lambda$, Sol: $\boxed{\{(-\lambda,\lambda,\lambda)|\lambda\in R\}}$

Si $\lambda = 1$, $M^* \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left\{\begin{array}{c} x+y+z=1 \\ y-z=1 \end{array}\right\}$ Si $z=\lambda$, Sol: $\boxed{\{(-2\lambda, 1+\lambda, \lambda) / \lambda \in R\}}$

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33) a) Si $A^2 = I \Rightarrow A \cdot A = I \Rightarrow A^{-1} \cdot A \cdot A = A^{-1} \cdot I \Rightarrow \boxed{A^{-1} = A}$

b) Como $A^2 = I \Rightarrow A^3 = A \Rightarrow A^4 = A^2 = I \Rightarrow \boxed{A^n = \begin{cases} A & si \quad n \quad es \quad impar \\ I & si \quad n \quad es \quad par \end{cases}}$

c) Si $A = \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \Rightarrow$ como $A^2 = I \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1+a \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$

$$\Rightarrow \left\{\begin{array}{c} 1=1 \\ 1+a=0 \\ 0=0 \\ a^2=1 \end{array}\right\} \Rightarrow \boxed{a=-1}$$

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34) a) $\left\{\begin{array}{cccc} (m+2)x & +(m-1)y & -z & = 3 \\ mx & -y & +z & = 2 \\ x & +my & -z & = 1 \end{array}\right. \Rightarrow$ Matrices: $M = \begin{pmatrix} m+2 & m-1 & -1 \\ m & -1 & 1 \\ 1 & m & -1 \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} m+2 & m-1 & -1 & 3 \\ m & -1 & 1 & 2 \\ 1 & m & -1 & 1 \end{array}\right)$

Si $m=1$, $M^* \approx \left(\begin{array}{ccc|c} 3 & 0 & -1 & 3 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{array}\right) \approx F_3 + F_2 \approx \left(\begin{array}{ccc|c} 3 & 0 & -1 & 3 \\ 1 & -1 & 1 & 2 \\ 2 & 0 & 0 & 3 \end{array}\right) \Rightarrow \left\{\begin{array}{c} 3x-z=3 \Rightarrow z=3x-3 \\ x-y+z=2 \\ 2x=3 \Rightarrow x=3/2 \end{array}\right\} \Rightarrow$

Solución : $\boxed{x=\frac{3}{2}}$, $\boxed{y=1}$, $\boxed{z=\frac{3}{2}}$

b) $|M| = \begin{vmatrix} m+2 & m-1 & -1 \\ m & -1 & 1 \\ 1 & m & -1 \end{vmatrix} = -m^2 - m = 0 \Rightarrow \left\langle \begin{array}{c} m=0 \\ m=-1 \end{array}\right.$

- <u>Caso 1</u>.- Si $m \neq -1, m \neq 0$, $|M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$.
- <u>Caso 2</u>.- Si $m=0$, $rang(M^*) = rang\left(\begin{array}{ccc|c} 2 & -1 & -1 & 3 \\ 0 & -1 & 1 & 2 \\ 1 & 0 & -1 & 1 \end{array}\right) = F_3 \leftrightarrow F_1 = rang\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 2 \\ 2 & -1 & -1 & 3 \end{array}\right) = F_3 - 2F_1 =$

$$= rang\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right) = F_3 - F_2 = rang\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$\Rightarrow rang(M) = 2 \neq 3 = rang(M^*) \Rightarrow SI$

- Caso 3.- Si $m=-1$, $rang(M^*)=rang\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ -1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 2 \end{array}\right)=\begin{array}{c} F_2+F_1 \\ F_3-F_1 \end{array}=rang\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & -3 & 0 & 3 \\ 0 & 1 & 0 & 2 \end{array}\right)=F_2 \leftrightarrow F_3=$

$$=rang\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & -3 & 0 & 3 \end{array}\right)=F_3+3F_2=rang\left(\begin{array}{ccc|c} 1 & -2 & -1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 9 \end{array}\right)\Rightarrow rang(M)=2\neq 3=rang(M^*)\Rightarrow SI$$

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35) $\begin{vmatrix} a^2 & ab & b^2 \\ 2a & a+b & 2b \\ 1 & 1 & 1 \end{vmatrix}=C_1-C_2=\begin{vmatrix} a^2-ab & ab & b^2 \\ a-b & a+b & 2b \\ 0 & 1 & 1 \end{vmatrix}=C_2-C_3=\begin{vmatrix} a^2-ab & ab-b^2 & b^2 \\ a-b & a-b & 2b \\ 0 & 0 & 1 \end{vmatrix}=$ desarrollando por la 3ª

fila $=\begin{vmatrix} a^2-ab & ab-b^2 \\ a-b & a-b \end{vmatrix}=C_1-C_2=\begin{vmatrix} a^2-2ab+b^2 & ab-b^2 \\ 0 & a-b \end{vmatrix}=\begin{vmatrix} (a-b)^2 & ab-b^2 \\ 0 & a-b \end{vmatrix}=(a-b)^3$

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36) Sea $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, si $B\cdot\begin{pmatrix} \lambda & 0 \\ 3 & 1 \end{pmatrix}=B\cdot\begin{pmatrix} 3 & 0 \\ 9 & 3 \end{pmatrix}\Rightarrow\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} \lambda & 0 \\ 3 & 1 \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 3 & 0 \\ 9 & 3 \end{pmatrix}\Rightarrow$

$\Rightarrow\begin{pmatrix} a\lambda+3b & b \\ c\lambda+3d & d \end{pmatrix}=\begin{pmatrix} 3a+9b & 3b \\ 3c+9d & 3d \end{pmatrix}\Rightarrow\left\{\begin{array}{c} a\lambda+3b=3a+9b \\ b=3b\Rightarrow b=0 \\ c\lambda+3d=3c+9d \\ d=3d\Rightarrow d=0 \end{array}\right\}\Rightarrow\left\{\begin{array}{c} a\lambda=3a \\ c\lambda=3c \end{array}\right\}\Rightarrow$ suponiendo $a\neq 0, c\neq 0 \Rightarrow \boxed{\lambda=3}$

Las matrices serán de la forma $\boxed{B=\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}}$

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37) a) $\left\{\begin{array}{rrrcl} 3x & +4y & +3z & = & 9 \\ mx & +2y & +z & = & 5 \\ x & +y & +z & = & 2 \end{array}\right.\Rightarrow$ Matrices asociadas: $M=\begin{pmatrix} 3 & 4 & 3 \\ m & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c} 3 & 4 & 3 & 9 \\ m & 2 & 1 & 5 \\ 1 & 1 & 1 & 2 \end{array}\right)$

$$|M|=\begin{vmatrix} 3 & 4 & 3 \\ m & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}=m-1=0\Rightarrow m=1$$

- Caso 1.- Si $m\neq 1$, , $|M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$.

- Caso 2.- Si $m=1$, $rang(M^*)=rang\left(\begin{array}{ccc|c} 3 & 4 & 3 & 9 \\ 1 & 2 & 1 & 5 \\ 1 & 1 & 1 & 2 \end{array}\right)=F_3\leftrightarrow F_1=rang\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 5 \\ 3 & 4 & 3 & 9 \end{array}\right)=\begin{array}{c} F_2-F_1 \\ F_3-3F_1 \end{array}=$

$$=rang\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 3 \end{array}\right)=F_3-F_2=rang\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

b) Si $m=1$, $M^* \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left\{\begin{array}{c} x+y+z=2 \\ y=3 \end{array}\right\}$ Si $z=\lambda$, Sol: $\boxed{\{(-1-\lambda, 3, \lambda) / \lambda \in R\}}$

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38) a) Sean : $\left\{\begin{array}{c} x = \text{precio billetes nacionales} \\ y = \text{precio billetes comunitarios} \\ z = \text{precio billetes no comunitarios} \end{array}\right\}$. Se plantea el siguiente sistema :

$$\left\{\begin{array}{llllll} 10x & +10y & +10z & = & 12000 \\ 10x & & +20z & = & 13000 \\ 10x & +10y & & = & 7000 \end{array}\right. \Rightarrow \left\{\begin{array}{llllll} x & +y & +z & = & 1200 \\ x & & +2z & = & 1300 \\ x & +y & & = & 700 \end{array}\right.$$

Matrices asociadas: $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 1 & 0 & 2 & 1300 \\ 1 & 1 & 0 & 700 \end{array}\right)$

Resolviendo: $M^* \approx F_i - F_1 \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1200 \\ 0 & -1 & 1 & 100 \\ 0 & 0 & -1 & -500 \end{array}\right) \Rightarrow \left\{\begin{array}{c} x+y+z=1200 \Rightarrow z=1200-400-500=300 \\ -y+500=100 \Rightarrow y=400 \\ -z=-500 \Rightarrow z=500 \end{array}\right\} \Rightarrow$

Solución: $\boxed{x=300€}$, $\boxed{y=400€}$, $\boxed{z=500€}$

b) Si el precio de los billetes nacionales bajan un 20%, costarán un 80%, es decir, $\frac{80}{100} \cdot x = 0.8 \cdot 300 = 240€$ por lo que hay que subir el precio de los billetes comunitarios: $300-240=60€$, luego el nuevo precio de estos sería $400+60=460€$, lo que supone una subida $400(1+t)=460 \Rightarrow t=0.15$. Subida del $\boxed{15\%}$

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39) Sean A y B matrices invertibles

a) Como $A+B=A\cdot B \Rightarrow A+B-A\cdot B=0 \Rightarrow B+A\cdot(I-B)=0 \Rightarrow B=-A\cdot(I-B)$, multiplicando por la inversa de $I-B$: $B\cdot(I-B)^{-1}=-A\cdot(I-B)\cdot(I-B)^{-1} \Rightarrow B\cdot(I-B)^{-1}=-A$, multiplicando de nuevo por la inversa de B, $B^{-1}\cdot B\cdot(I-B)^{-1}=-B^{-1}\cdot A \Rightarrow (I-B)^{-1}=-B^{-1}\cdot A$

b) Si $A=\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$. Sea $B=\begin{pmatrix} x & y \\ z & t \end{pmatrix}$, como $A+B=A\cdot B \Rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}+\begin{pmatrix} x & y \\ z & t \end{pmatrix}=\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} x & y \\ z & t \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}+\begin{pmatrix} x & y \\ z & t \end{pmatrix}=\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} x & y \\ z & t \end{pmatrix} \Rightarrow \begin{pmatrix} -1+x & 1+y \\ 2+z & -1+t \end{pmatrix}=\begin{pmatrix} -x+z & -y+t \\ 2x-z & 2y-t \end{pmatrix} \Rightarrow \text{Igualando:}$$

$$\Rightarrow \left\{\begin{array}{c} -1+x=-x+z \\ 1+y=-y+t \\ 2+z=2x-z \\ -1+t=2y-t \end{array}\right\} \Rightarrow \Rightarrow \left\langle\begin{array}{l} \left\{\begin{array}{c} 2x=z+1 \\ 2x=2z+2 \end{array}\right\} \Rightarrow z+1=2z+2 \Rightarrow \left\langle\begin{array}{c} z=-1 \\ x=0 \end{array}\right. \\ \left\{\begin{array}{c} 2y=t-1 \\ 2y=2t-1 \end{array}\right\} \Rightarrow t-1=2t-1 \Rightarrow \left\langle\begin{array}{c} t=0 \\ y=-1/2 \end{array}\right. \end{array}\right. \Rightarrow \boxed{B=\begin{pmatrix} 0 & -1/2 \\ -1 & 0 \end{pmatrix}}$$

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40) a) $\begin{cases} (1-a)x & -2y & +4z & = & 0 \\ x & -(1+a)y & +z & = & 0 \\ -x & +ay & -z & = & 0 \end{cases} \Rightarrow$ Matrices asociadas: $M = \begin{pmatrix} 1-a & -2 & 4 \\ 1 & -(1-a) & 1 \\ -1 & a & -1 \end{pmatrix}$ y

$M^* = \left(\begin{array}{ccc|c} 1-a & -2 & 4 & 0 \\ 1 & -(1-a) & 1 & 0 \\ -1 & a & -1 & 0 \end{array}\right)$. Como es un sistema homogéneo $rang(M) = rang(M^*)$

$|M| = \begin{vmatrix} 1-a & -2 & 4 \\ 1 & -(1-a) & 1 \\ -1 & a & -1 \end{vmatrix} = -a-3 = 0 \Rightarrow a = -3$

- Caso 1.- Si $a \neq -3$, , $|M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$.
- Caso 2.- Si $a = -3$, $rang(M) = rang\begin{pmatrix} 2 & -2 & 4 \\ 1 & 2 & 1 \\ -1 & -3 & -1 \end{pmatrix} = F_3 \leftrightarrow F_1 = rang\begin{pmatrix} -1 & -3 & -1 \\ 1 & 2 & 1 \\ 2 & -2 & 4 \end{pmatrix} = \dfrac{F_2+F_1}{F_3+2F_1} =$

$= rang\begin{pmatrix} -1 & -3 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow rang(M) = 2 = rang(M^*) \Rightarrow SCI$

b) Sea $a = -3$, $M^* \approx \begin{pmatrix} -1 & -3 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{Bmatrix} -x-3y-z=0 \\ -y=0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x=-z \\ y=0 \end{Bmatrix}$. Si $z = \lambda$, Sol: $\boxed{\{(-\lambda, 0, \lambda) / \lambda \in R\}}$

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41) a) $A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & -1 \\ 5 & -1 & 2 \end{pmatrix} \Rightarrow \boxed{A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix}}$

b) Si $A \cdot X \cdot A^t = B \Rightarrow A^{-1} \cdot A \cdot X \cdot A^t = A^{-1} \cdot B \Rightarrow X \cdot A^t = A^{-1} \cdot B \Rightarrow X \cdot A^t \cdot (A^t)^{-1} = A^{-1} \cdot B \cdot (A^t)^{-1} \Rightarrow$

$\Rightarrow X = A^{-1} \cdot B \cdot (A^{-1})^t$. Es decir,

$X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -2 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & -2 \\ 1 & -3 & -4 \\ -2 & -4 & 3 \end{pmatrix}}$

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42) a) Sea el sistema: $\begin{Bmatrix} x+2y=1 \\ 3x-y=2 \end{Bmatrix}$. Podemos añadir cualquier ecuación equivalente a una de ellas, por ejemplo:

$\boxed{2x+4y=2}$ (dos veces la 1ª ecuación)

b) $\begin{Bmatrix} 2x+2y-z=1 \\ x+y+2z=1 \end{Bmatrix}$. Tomamos, por ejemplo, "dos veces 1ª - 2ª", o sea: $\boxed{3x+3y-4z=1}$

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43) a) $B=\begin{pmatrix}1&1&2\\1&1&-1\\0&1&3\end{pmatrix}\Rightarrow \boxed{B^{-1}=\begin{pmatrix}4/3&-1/3&-1\\-1&1&1\\1/3&-1/3&0\end{pmatrix}}$

b) $B\cdot X=A\Rightarrow B^{-1}\cdot B\cdot X=B^{-1}\cdot A\Rightarrow X=B^{-1}\cdot A$. Es decir:

$$X=\begin{pmatrix}4/3&-1/3&-1\\-1&1&1\\1/3&-1/3&0\end{pmatrix}\cdot\begin{pmatrix}1&2&0\\0&1&2\\0&2&3\end{pmatrix}=\boxed{\begin{pmatrix}4/3&1/3&-11/3\\-1&1&5\\1/3&1/3&-2/3\end{pmatrix}}$$

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44) a) $A^2=\begin{pmatrix}0&0\\0&0\end{pmatrix}\Rightarrow |A^2|=0\Rightarrow|A\cdot A|=0\Rightarrow|A|\cdot|A|=0\Rightarrow|A|^2=0\Rightarrow \boxed{|A|=0}$

b) $\left[\begin{pmatrix}3&-4\\1&-1\end{pmatrix}-k\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\right]^2=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. Por el apartado anterior $\left|\begin{pmatrix}3&-4\\1&-1\end{pmatrix}-k\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}\right|=0\Rightarrow$

$$\begin{vmatrix}3-k&-4\\1&-1-k\end{vmatrix}=0\Rightarrow k^2-2k+1=0\Rightarrow(k-1)^2=0\Rightarrow\boxed{k=1}$$

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45) a) $\begin{cases}\lambda x+3y+z=\lambda\\x+\lambda y+\lambda z=1\\x+y-z=1\end{cases}\Rightarrow$ Matrices asociadas: $M=\begin{pmatrix}\lambda&3&1\\1&\lambda&\lambda\\1&1&-1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c}\lambda&3&1&\lambda\\1&\lambda&\lambda&1\\1&1&-1&1\end{array}\right)$

$$|M|=\begin{vmatrix}1&1&\lambda\\0&1&-1\\1&\lambda&1\end{vmatrix}=0\Rightarrow-2\lambda^2+2\lambda+4=0\Rightarrow\left\langle\begin{array}{l}\lambda=2\\\lambda=-1\end{array}\right.$$

- Caso 1.- Si $\lambda\neq2,\ \lambda\neq-1,\ |M|\neq0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- Caso 2.- Si $\lambda=2$,. $rang(M^*)=rang\left(\begin{array}{ccc|c}2&3&1&2\\1&2&2&1\\1&1&-1&1\end{array}\right)=F_3\leftrightarrow F_1=rang\left(\begin{array}{ccc|c}1&1&-1&1\\1&2&2&1\\2&3&1&2\end{array}\right)=\begin{array}{c}F_2-F_1\\F_3-2F_1\end{array}=$

$$=rang\left(\begin{array}{ccc|c}1&1&-1&1\\0&1&3&0\\0&1&3&0\end{array}\right)=F_3-F_2=rang\left(\begin{array}{ccc|c}1&1&-1&1\\0&1&3&0\\0&0&0&0\end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

- Caso 3.- Si $\lambda=-1$,. $rang(M^*)=rang\left(\begin{array}{ccc|c}-1&3&1&-1\\1&-1&-1&1\\1&1&-1&1\end{array}\right)=\begin{array}{c}F_2+F_1\\F_3+F_1\end{array}=rang\left(\begin{array}{ccc|c}-1&3&1&-1\\0&2&0&0\\0&4&0&0\end{array}\right)=F_3-2F_2=$

$$=rang\left(\begin{array}{ccc|c}-1&3&1&-1\\0&2&0&0\\0&0&0&0\end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

b) Si $\lambda=2$, $M^*\approx\left(\begin{array}{ccc|c}1&1&-1&1\\0&1&3&0\\0&0&0&0\end{array}\right)\Rightarrow\left.\begin{cases}x+y-z=1\\y+3z=0\end{cases}\right\}\Rightarrow\left.\begin{cases}x=1-y+z=1+4z\\y=-3z\end{cases}\right\}$ Si $z=t$, Sol: $\boxed{\{(1+4t,-3t,t)\mid t\in R\}}$

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46) a) $\begin{cases}(m-1)x & +y & +z & = & 3\\ mx & +(m-1)y & +3z & = & 2m-1\\ x & +2y & +(m-2)z & = & 4\end{cases}\Rightarrow$ Matrices asociadas : $M=\begin{pmatrix}m-1&1&1\\m&m-1&3\\1&2&m-2\end{pmatrix}$ y

$$M^*=\left(\begin{array}{ccc|c}m-1&1&1&3\\m&m-1&3&2m-1\\1&2&m-2&4\end{array}\right)$$

$$|M|=\begin{vmatrix}m-1&1&1\\m&m-1&3\\1&2&m-2\end{vmatrix}=m^3-5m^2+2m+8=0\Rightarrow\left\langle\begin{array}{l}m=-1\\m=2\\m=4\end{array}\right.$$

- Caso 1.- Si $m\neq-1,$, $m\neq2$, $m\neq4$, $|M|\neq0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$.
- Caso 2.- Si $m=-1$, $rang(M^*)=rang\left(\begin{array}{ccc|c}-2&1&1&3\\-1&-2&3&-3\\1&2&-3&4\end{array}\right)=F_3\leftrightarrow F_1=rang\left(\begin{array}{ccc|c}1&2&-3&4\\-1&-2&3&-3\\-2&1&1&3\end{array}\right)=$

$$=\begin{array}{c}F_2+F_1\\F_3+2F_1\end{array}=rang\left(\begin{array}{ccc|c}1&2&-3&4\\0&0&0&1\\0&5&-5&11\end{array}\right)\Rightarrow rang(M)=2\neq3=rang(M^*)\Rightarrow SI$$

- Caso 3.- Si $m=2$, $rang(M^*)=rang\left(\begin{array}{ccc|c}1&1&1&3\\2&1&3&3\\1&2&0&4\end{array}\right)=\begin{array}{c}F_2-2F_1\\F_3-F_1\end{array}=rang\left(\begin{array}{ccc|c}1&1&1&3\\0&-1&1&-3\\0&1&-1&1\end{array}\right)=F_3+F_2=$

$$=rang\left(\begin{array}{ccc|c}1&1&1&3\\0&-1&1&-3\\0&0&0&-2\end{array}\right)\Rightarrow rang(M)=2\neq3=rang(M^*)\Rightarrow SI$$

- Caso 4.- Si $m=4$, $rang(M^*)=rang\left(\begin{array}{ccc|c}3&1&1&3\\4&3&3&7\\1&2&2&4\end{array}\right)=F_3\leftrightarrow F_1=rang\left(\begin{array}{ccc|c}1&2&2&4\\4&3&3&7\\3&1&1&3\end{array}\right)=\begin{array}{c}F_2-4F_1\\F_3-3F_1\end{array}=$

$$=rang\left(\begin{array}{ccc|c}1&2&2&4\\0&-5&-5&-9\\0&-5&-5&-9\end{array}\right)=F_3-F_2=rang\left(\begin{array}{ccc|c}1&2&2&4\\0&-5&-5&-9\\0&0&0&0\end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

b) Si $m=4$, $M^*\approx\left(\begin{array}{ccc|c}1&2&2&4\\0&-5&-5&-9\\0&0&0&0\end{array}\right)\Rightarrow\left.\begin{cases}x+2y+2z=4\\-5y-5z=-9\end{cases}\right\}\Rightarrow\left.\begin{cases}x=2/5\\y=9/5-z\end{cases}\right\}$ Si $z=\lambda$, Sol: $\boxed{\left\{\left(\frac{2}{5},\frac{9}{5}-\lambda,\lambda\right)\middle|\lambda\in R\right\}}$

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47) a) $\begin{cases} x+2y+3z=1 \\ 2x+y-z=2 \end{cases}$ Matrices asociadas: $M=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 2 \end{array}\right)$

$M^* \approx \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 2 \end{array}\right) = F_2 - 2F_1 \approx \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 0 \end{array}\right) \Rightarrow \left\{\begin{array}{l} x+2y+3z=1 \Rightarrow x=1-2y-3z \\ -3y-7z=0 \Rightarrow y=(-3/7)z \end{array}\right\} \Rightarrow$ Si $z=\lambda$,

Sol: $\boxed{\left\{\left(1+\frac{5\lambda}{3}, \frac{-7\lambda}{3}, \lambda\right) \middle| \lambda \in R\right\}}$

b) Consideremos el sistema: $\begin{cases} x+2y+3z=1 \\ 2x+y-z=2 \\ 5x+y+\alpha z=\beta \end{cases}$. Matrices asociadas: $M=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 5 & 1 & \alpha \end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & 1 & \alpha & \beta \end{array}\right)$

Si el sistema tiene que ser compatible indeterminado, $rang(M)=2=rang(M^*)$.

Como $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \neq 0 \Rightarrow rang(M) \geq 2$. Para que $rang(M)=2, |M|=0 \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & -1 \\ 5 & 1 & \alpha \end{vmatrix} = 0 \Rightarrow -3\alpha-18=0 \Rightarrow$

$\Rightarrow \boxed{\alpha=-6}$

Sustituyendo el valor anterior, $rang(M^*) = rang\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & 1 & -6 & \beta \end{array}\right) = \begin{matrix} F_2-2F_1 \\ F_3-5F_1 \end{matrix} = rang\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & -9 & -21 & \beta-5 \end{array}\right) =$

$= F_3 - 3F_2 = rang\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -7 & 0 \\ 0 & 0 & 0 & \beta-5 \end{array}\right)$ $rang(M^*)=2 \Rightarrow \beta-5=0 \Rightarrow \boxed{\beta=5}$

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48) Si $A^{-1} \cdot X \cdot A = B \Rightarrow A \cdot A^{-1} \cdot X \cdot A = A \cdot B \Rightarrow X \cdot A = A \cdot B \Rightarrow X \cdot A \cdot A^{-1} = A \cdot B \cdot A^{-1} \Rightarrow X = A \cdot B \cdot A^{-1}$

Como $A=\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} \Rightarrow A^{-1}=\begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}$. Así $X=\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix} \Rightarrow \boxed{X=\begin{pmatrix} 9 & 11 \\ -6 & -7 \end{pmatrix}}$

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49) Si $A=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow A^2=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

a) Si $A^2=\alpha A+\beta I \Rightarrow \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha+\beta & 2\alpha \\ 0 & \alpha+\beta \end{pmatrix} \Rightarrow \left\{\begin{array}{l} \alpha+\beta=1 \\ 2\alpha=4 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} \alpha=2 \\ \beta=-1 \end{array}}$

b) Del apartado anterior, $A^2=2A-I$. Por otra parte: $A^5=A^4 \cdot A=(A^2)^2 \cdot A=(2A-I)^2 \cdot A=$ como las matrices A e I conmutan $=(4A^2+I^2-4A) \cdot A=$ (como $I^2=I$) $=(4A^2+I-4A) \cdot A=$ sustituyendo la expresión de $A^2=[4(2A-I)+I-4A] \cdot A=(8A-4I+I) \cdot A=(4A-3I) \cdot A=4A^2-3A=$ sustituyendo de nuevo $A^2=4(2A-I)-3A=8A-4I-3A=5A-4I=5\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}-4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 5-4 & 10 \\ 0 & 5-4 \end{pmatrix} \Rightarrow$

$$\Rightarrow \boxed{A^5 = \begin{pmatrix} 1 & 10 \\ 0 & 1 \end{pmatrix}}$$

c) Si se tiene que cumplir que $(A-X)\cdot(A+X) = A^2 - X^2 \Rightarrow A^2 + AX - XA - X^2 = A^2 - X^2 \Rightarrow AX - XA = 0$

es decir, $AX = XA \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2a+b \\ c & 2c+d \end{pmatrix} \Rightarrow$

$$\Rightarrow \left\{\begin{array}{c} a+2c = a \Rightarrow c = 0 \\ b+2d = 2a+b \Rightarrow d = a \\ c = c \\ d = 2c+2d \end{array}\right\} \Rightarrow \boxed{X = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}}$$

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50) a) $A^2 = \begin{pmatrix} 0 & k & t \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & k & t \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & k^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow A^3 = A^2\cdot A = \begin{pmatrix} 0 & 0 & k^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & k & t \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix} = 0 \Rightarrow \boxed{A^{10} = 0}$

b) $\boxed{B^{-1} = \begin{pmatrix} 1 & -k & k^2 - t \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{pmatrix}}$

c) Si $k = 0$, $B = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow B^2 = \begin{pmatrix} 1 & 0 & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow B^3 = \begin{pmatrix} 1 & 0 & 3t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \ldots\ldots \Rightarrow \boxed{B^{10} = \begin{pmatrix} 1 & 0 & 10t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

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51) $\left\{\begin{array}{llll} x & +ky & -z & = 0 \\ kx & -y & +z & = 0 \\ (k+1)x & +y & & = 0 \end{array}\right. \Rightarrow$ Matrices asociadas: $M = \begin{pmatrix} 1 & k & -1 \\ k & -1 & 1 \\ k+1 & 1 & 0 \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} 1 & k & -1 & 0 \\ k & -1 & 1 & 0 \\ k+1 & 1 & 0 & 0 \end{array}\right)$.

Como es un sistema homogéneo $rang(M) = rang(M^*)$

$|M| = \begin{vmatrix} 1 & k & -1 \\ k & -1 & 1 \\ k+1 & 1 & 0 \end{vmatrix} = k^2 - 2k - 2 = 0 \Rightarrow \left\langle\begin{array}{l} k = 2 \\ k = -1 \end{array}\right.$. Para que no tenga la solución trivial, $|M| = 0$, luego

$\boxed{k = 2}$, $\boxed{k = -1}$

- Si $k = -1$, $M = \begin{pmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \approx F_2 + F_1 \approx \begin{pmatrix} 1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \approx F_2 + 2F_3 \approx \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow$

$$\Rightarrow \left\{\begin{array}{c} x - y - z = 0 \Rightarrow x = z \\ y = 0 \end{array}\right\} \Rightarrow \text{ Si } z = \lambda, \text{ Sol: } \boxed{\{(\lambda, 0, \lambda) | \lambda \in R\}}$$

- Si $k = 2$, $M = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix} \approx \begin{array}{c} F_2 - 2F_1 \\ F_3 - 3F_1 \end{array} \approx \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{pmatrix} \approx F_3 - F_2 \approx \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$

$$\Rightarrow \left\{\begin{matrix} x+2y-z=0 \Rightarrow x=-z/5 \\ 5y-3z=0 \Rightarrow y=3z/5 \end{matrix}\right\} \Rightarrow \text{ Si } z=\lambda \text{, Sol: } \boxed{\left\{\left(\frac{-\lambda}{5}, \frac{3\lambda}{5}, \lambda\right) \middle| \lambda \in R\right\}}$$

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52) Si $AP = PA \Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a+2c & b+2d \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2a+b \\ c & 2c+d \end{pmatrix} \Rightarrow$

$$\Rightarrow \left\{\begin{matrix} a+2c=a \Rightarrow c=0 \\ b+2d=2a+b \Rightarrow d=a \\ c=c \\ d=2c+2d \end{matrix}\right\} \Rightarrow \boxed{X = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}}$$

---

53) a) Calculamos $|M| = \begin{vmatrix} 2 & 1 & -a \\ 2a & 1 & -1 \\ 2 & a & 1 \end{vmatrix} = 0 \Rightarrow -2a^3+2a=0 \Rightarrow \left\langle \begin{matrix} a=0 \\ a=1 \\ a=-1 \end{matrix} \right.$

- <u>Caso 1</u>.- Si $a \neq 0, , a \neq 1\ a \neq -1,\ |M| \neq 0 \Rightarrow rang(M)=3$
- <u>Caso 2</u>.- Si $a=0,\ rang(M) = rang\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{pmatrix} = F_3 - F_1 = rang\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = F_3 + F_2 =$

$$= rang\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = 2$$

- <u>Caso 3</u>.- Si $a=1,\ rang(M) = rang\begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{matrix} F_2 - F_1 \\ F_3 - F_1 \end{matrix} = rang\begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2$
- <u>Caso 4</u>.- Si $a=-1,\ rang(M) = rang\begin{pmatrix} 2 & 1 & 1 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} = \begin{matrix} F_2 + F_1 \\ F_3 - F_1 \end{matrix} = rang\begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2$

b) Existirá inversa si $a \neq 0, , a \neq 1\ a \neq -1,$

Si $a=2 \Rightarrow M = \begin{pmatrix} 2 & 1 & -2 \\ 4 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix} \Rightarrow \boxed{M^{-1} = \begin{pmatrix} -1/4 & 5/12 & -1/12 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/6 & 1/6 \end{pmatrix}}$

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54) a) $|A| = \begin{vmatrix} 3 & 1 \\ -8 & -3 \end{vmatrix} = -1$; $A^2 = \begin{pmatrix} 3 & 1 \\ -8 & -3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ -8 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \Rightarrow |A^2| = 1 = (-1)^2$

$$|A+I| = \left|\begin{pmatrix} 3 & 1 \\ -8 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = \begin{vmatrix} 3+1 & 1+0 \\ -8+0 & -3+1 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ -8 & -2 \end{vmatrix} = 0$$

$$|A|+|I|=\begin{vmatrix}3 & 1\\ -8 & -3\end{vmatrix}+\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=-1+1=0$$

b) $|M^2|=|M\cdot M|=|M|\cdot|M|=|M|^2$

c) Sea $M=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\Rightarrow |M|=ad-bc$. Además $|I|=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1$

$$|M+I|=\left|\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right|=\left|\begin{pmatrix}a+1 & b\\ c & d+1\end{pmatrix}\right|=(a+1)(d+1)-bc$$

Si $|M+I|=|M|+|I|\Rightarrow ad-bc+1=(a+1)(d+1)-bc\Rightarrow ad-bc+1=ad+a+d+1-bc\Rightarrow a+d=0\Rightarrow$

$\Rightarrow d=-a\Rightarrow \boxed{M=\begin{pmatrix}a & b\\ c & -a\end{pmatrix}}$

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55) a) $\begin{cases}x+2y-3z=0\\ 2x+3y-z=5\end{cases}$ Matrices asociadas: $M=\begin{pmatrix}1 & 1 & -3\\ 2 & 3 & -1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c}1 & 1 & -3 & 0\\ 2 & 3 & -1 & 5\end{array}\right)$

$$M^*\approx\left(\begin{array}{ccc|c}1 & 1 & -3 & 0\\ 2 & 3 & -1 & 5\end{array}\right)=F_2-2F_1\approx\left(\begin{array}{ccc|c}1 & 1 & -3 & 0\\ 0 & 1 & 5 & 5\end{array}\right)\Rightarrow\left\{\begin{array}{c}x+y-3z=0\Rightarrow x=-y+3z\\ y+5z=5\Rightarrow y=5-5z\end{array}\right\}\Rightarrow \text{ Si } z=\lambda\text{ ,}$$

Sol: $\boxed{\{(8\lambda,5-5\lambda,\lambda)/\lambda\in R\}}$

b) Se platea el sistema: $\begin{cases}x+y-3z=0\\ 2x+3y-z=5\\ x+y+z=4\end{cases}\Rightarrow$ Matriz asociada: $M^*=\left(\begin{array}{ccc|c}1 & 1 & -3 & 0\\ 2 & 3 & -1 & 5\\ 1 & 1 & 1 & 4\end{array}\right)$

$$M^*=\left(\begin{array}{ccc|c}1 & 1 & -3 & 0\\ 2 & 3 & -1 & 5\\ 1 & 1 & 1 & 4\end{array}\right)\approx\begin{matrix}F_2-2F_1\\ F_3-F_1\end{matrix}\approx\left(\begin{array}{ccc|c}1 & 1 & -3 & 0\\ 0 & 1 & 5 & 5\\ 0 & 0 & 4 & 4\end{array}\right)\Rightarrow\left\{\begin{array}{c}x+y-3z=0\Rightarrow x=3z-y=3\\ y+5z=5\Rightarrow y=5-5z=0\\ 4z=4\Rightarrow z=1\end{array}\right\}\Rightarrow$$

Solución: $\boxed{x=3}$, $\boxed{y=0}$, $\boxed{z=1}$

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56) Si $A=\begin{pmatrix}a & a\\ 0 & b\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}a & a\\ 0 & b\end{pmatrix}\cdot\begin{pmatrix}a & a\\ 0 & b\end{pmatrix}=\begin{pmatrix}a^2 & a^2+ab\\ 0 & b^2\end{pmatrix}$

a) Si $A^2=A\Rightarrow\begin{pmatrix}a^2 & a^2+ab\\ 0 & b^2\end{pmatrix}=\begin{pmatrix}a & a\\ 0 & b\end{pmatrix}\Rightarrow\left\{\begin{array}{c}a^2=a\Rightarrow a=\left\langle\begin{array}{l}1\\ 0\end{array}\right.\\ a^2+ab=a\Rightarrow a(a+b)=a\\ b^2=b\Rightarrow b=\left\langle\begin{array}{l}1\\ 0\end{array}\right.\end{array}\right\}$

<u>Caso 1</u>.- Si $a=0\Rightarrow b=1\Rightarrow \boxed{A=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}}$

Caso 2.- Si $a \neq 0 \Rightarrow a=1 \Rightarrow a+b=1 \Rightarrow b=0 \Rightarrow \boxed{A=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}}$

b) Caso 1.- Si $A=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \Rightarrow A^2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=A \Rightarrow A^3=A \Rightarrow ...... \Rightarrow A^{10}=A$

Así, $M=A+A^2+A^3+...+A^{10}=A+A+A+.....+A=10A \Rightarrow \boxed{M=\begin{pmatrix}0 & 0\\ 0 & 10\end{pmatrix}}$

Caso 2.- Si $A=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} \Rightarrow A^2=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\cdot\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}=A \Rightarrow A^3=A \Rightarrow ...... \Rightarrow A^{10}=A$

Así, $M=A+A^2+A^3+...+A^{10}=A+A+A+.....+A=10A \Rightarrow \boxed{M=\begin{pmatrix}10 & 10\\ 0 & 0\end{pmatrix}}$

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57) $|M|=\begin{vmatrix}m & m-1 & m(m-1)\\ m & 1 & m\\ m & 1 & m-1\end{vmatrix}=m(m-2)=0 \Rightarrow \left\langle \begin{matrix}m=0\\ m=2\end{matrix}\right.$

- Caso 1.- Si $m \neq 0,\, , m \neq 2,\ |M| \neq 0 \Rightarrow rang(M)=3$
- Caso 2.- Si $m=0,\ rang\begin{pmatrix}0 & -1 & 0\\ 0 & 1 & 0\\ 0 & 1 & -1\end{pmatrix}=F_i+F_1=rang\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}=2$
- Caso 3.- Si $m=2,\ rang\begin{pmatrix}2 & 1 & 2\\ 2 & 1 & 2\\ 2 & 1 & 1\end{pmatrix}=F_i-F_1=rang\begin{pmatrix}2 & 1 & 2\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}=2$

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58) Si $X\cdot A\cdot X^{-1}=B \Rightarrow X\cdot A\cdot X^{-1}\cdot X=B\cdot X \Rightarrow X\cdot A=B\cdot X$

Sea $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow \begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}8 & -9\\ 6 & -7\end{pmatrix}\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow \begin{pmatrix}2a & -b\\ 2c & -d\end{pmatrix}=\begin{pmatrix}8a-9c & 8b-9d\\ 6a-7c & 6b-7d\end{pmatrix} \Rightarrow$

$$\Rightarrow \left\{\begin{matrix}2a=8a-9c\\ 2c=6a-7c\\ -b=8b-9d\\ -d=6b-7d\end{matrix}\right\} \Rightarrow \left\{\begin{matrix}2a-3c=0 \Rightarrow c=2a/3\\ b-d=0 \Rightarrow d=b\end{matrix}\right\} \Rightarrow \boxed{X=\begin{pmatrix}a & b\\ 3a/2 & b\end{pmatrix}}$$

---

59) a) Si $A\cdot B=B\cdot A \Rightarrow \begin{pmatrix}5 & 2 & 0\\ 2 & 5 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}a & b & 0\\ c & c & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}a & b & 0\\ c & c & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}5 & 2 & 0\\ 2 & 5 & 0\\ 0 & 0 & 1\end{pmatrix} \Rightarrow$

$$\Rightarrow\begin{pmatrix}5a+2c & 2c+5b & 0\\ 5c+2a & 2b+5c & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}5a+2b & 2a+5b & 0\\ 7c & 7c & 0\\ 0 & 0 & 1\end{pmatrix}\Rightarrow\begin{Bmatrix}5a+2c=5a+2b\\ 2c+5b=2a+5b\\ 5c+2a=7c\\ 2b+5c=7c\end{Bmatrix}\Rightarrow\boxed{a=b=c}$$

b) Sea $a=b=c=1\Rightarrow B=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}\Rightarrow B^2=\begin{pmatrix}2&2&0\\2&2&0\\0&0&1\end{pmatrix}\Rightarrow B^3=\begin{pmatrix}2^2&2^2&0\\2^2&2^2&0\\0&0&1\end{pmatrix}\Rightarrow B^4=\begin{pmatrix}2^3&2^3&0\\2^3&2^3&0\\0&0&1\end{pmatrix}\Rightarrow.....$

$$\Rightarrow\boxed{B^{10}=\begin{pmatrix}2^9&2^9&0\\2^9&2^9&0\\0&0&1\end{pmatrix}}$$

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60) $\begin{cases}x+(k+1)y+2z=-1\\ kx+y+z=k\\ (k-1)x-2y-z=k+1\end{cases}\Rightarrow$ Matrices: $M=\begin{pmatrix}1&k+1&2\\k&1&1\\k-1&-2&-1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c}1&k+1&2&-1\\k&1&1&k\\k-1&-2&-1&k+1\end{array}\right)$.

$$|M|=\begin{vmatrix}1&k+1&2\\k&1&1\\k-1&-2&-1\end{vmatrix}=2k^2-5k+2=0\Rightarrow\left\langle\begin{matrix}k=2\\k=1/2\end{matrix}\right. .$$

- Caso 1.- Si $k\neq 2,\ ,k\neq 1/2,\ |M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- Caso 2.- Si $k=2,\ rang(M)=rang\left(\begin{array}{ccc|c}1&3&2&-1\\2&1&1&2\\1&-2&-1&3\end{array}\right)=\begin{matrix}F_2-2F_1\\F_3-F_1\end{matrix}=rang\left(\begin{array}{ccc|c}1&3&2&-1\\0&-5&-3&4\\0&-5&-3&4\end{array}\right)=F_3-F_2=$

$$=rang\left(\begin{array}{ccc|c}1&3&2&-1\\0&-5&-3&4\\0&0&0&0\end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

- Caso 3.- Si $k=1/2,\ rang(M)=rang\left(\begin{array}{ccc|c}1&3/2&2&-1\\1/2&1&1&1/2\\-1/2&-1&-1&3/2\end{array}\right)=\begin{matrix}2F_{21}\\2F_3\end{matrix}=rang\left(\begin{array}{ccc|c}1&3/2&2&-1\\1&2&2&1\\-1&-2&-2&3\end{array}\right)=$

$$=F_3+F_2=rang\left(\begin{array}{ccc|c}1&3/2&2&-1\\1&2&2&1\\0&0&0&4\end{array}\right)\Rightarrow rang(M)=2\neq 3=rang(M^*)\Rightarrow SI$$

b) Si $k=2$, $\Rightarrow\begin{Bmatrix}x+3y+2z=-1\\-5y-3z=4\end{Bmatrix}\Rightarrow$ Si $z=\lambda$, Sol: $\boxed{\left\{\left(\frac{7-\lambda}{5},\frac{-3\lambda-4}{5},\lambda\right)\middle|\lambda\in R\right\}}$

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61) $A^2=\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}\cdot\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I$

$B\cdot A=\begin{pmatrix}0&0&-2\\0&-2&0\\-2&0&0\end{pmatrix}\cdot\begin{pmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{pmatrix}=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}=2I$

Si $X\cdot A^2+B\cdot A=A^2\Rightarrow X\cdot I+2I=I\Rightarrow X=I-2I\Rightarrow \boxed{X=-I}$

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62) a) $\begin{cases}x+2y-3z=3\\2x+3y+z=5\end{cases}$ Matrices asociadas: $M=\begin{pmatrix}1&2&-3\\2&3&1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c}1&2&-3&3\\2&3&1&5\end{array}\right)$

$rang(M^*)=F_2-2F_1=rang\left(\begin{array}{ccc|c}1&2&-3&3\\0&-1&7&-1\end{array}\right)=2=rang(M)\Rightarrow SCI$

Sea $ax+y+bz=1$, con $\begin{cases}x+2y-3z=3\\2x+3y+z=5\\ax+y+bz=1\end{cases}$ SCI, con $rang\begin{pmatrix}1&2&-3\\2&3&1\\a&1&b\end{pmatrix}=2=rang\left(\begin{array}{ccc|c}1&2&-3&3\\2&3&1&5\\a&1&b&1\end{array}\right)$

Como $rang\begin{pmatrix}1&2&-3\\2&3&1\end{pmatrix}=2\Rightarrow\begin{vmatrix}1&2&-3\\2&3&1\\a&1&b\end{vmatrix}=0\Rightarrow b-11a+7=0\Rightarrow b=11a-7$

Sustituyendo: $rang\left(\begin{array}{ccc|c}1&2&-3&3\\2&3&1&5\\a&1&11a-7&1\end{array}\right)=\begin{matrix}F_2-2F_1\\F_3-aF_1\end{matrix}=rang\left(\begin{array}{ccc|c}1&2&-3&3\\0&-1&7&-1\\0&1-2a&14a-7&1-3a\end{array}\right)=F_3+(1-2a)F_2=$

$=rang\left(\begin{array}{ccc|c}1&2&-3&3\\0&-1&7&-1\\0&0&0&-a\end{array}\right)=2\Rightarrow -a=0\Rightarrow\boxed{a=0}$

Sustituyendo: $\boxed{b=7}$

b) $\begin{cases}x+2y-3z=3\\2x+3y+z=5\\x+y+z=4\end{cases}$, $M^*=\left(\begin{array}{ccc|c}1&2&-3&3\\2&3&1&5\\1&1&1&4\end{array}\right)\approx\begin{matrix}F_2-2F_1\\F_3-F_1\end{matrix}\approx\left(\begin{array}{ccc|c}1&2&-3&3\\0&-1&7&-1\\0&-1&4&1\end{array}\right)\approx F_3-F_2\approx\left(\begin{array}{ccc|c}1&2&-3&3\\0&-1&7&-1\\0&0&-3&2\end{array}\right)$

Así $\left\{\begin{array}{c}x+2y-3z=3\Rightarrow x=3-2y+3z=3+22/3-2=25/3\\-y+7z=-1\Rightarrow -y-14/3=-1\Rightarrow y=-11/3\\-3z=2\Rightarrow z=-2/3\end{array}\right\}$. Solución : $\boxed{x=\frac{25}{3}}$, $\boxed{y=-\frac{11}{3}}$, $\boxed{z=-\frac{2}{3}}$

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63) a) $\begin{cases}x-ay=2\\ax-y=a+1\end{cases}$ Matrices $M^*=\left(\begin{array}{cc|c}1&-a&2\\a&-1&a+1\end{array}\right)$; $|M|=\begin{vmatrix}1&-a\\a&-1\end{vmatrix}=a^2-1=0\Rightarrow a=\pm1$

- <u>Caso 1</u>.- Si $a\neq1\ a\neq-1$, $|M|\neq0\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCD$

- Caso 2.- Si $a=1$, $rang(M^*)=rang\left(\begin{array}{cc|c}1 & -1 & 2\\ 1 & -1 & 2\end{array}\right)=F_2-F_1=rang\left(\begin{array}{cc|c}1 & -1 & 2\\ 0 & 0 & 0\end{array}\right)\Rightarrow rang(M)=rang(M^*)=2$

  $\Rightarrow SCI$

- Caso 3.- Si $a=-1$, $rang(M^*)=rang\left(\begin{array}{cc|c}1 & 1 & 2\\ -1 & -1 & 0\end{array}\right)=F_2+F_1=rang\left(\begin{array}{cc|c}1 & 1 & 2\\ 0 & 0 & 2\end{array}\right)\Rightarrow$

  $rang(M)=1\neq 2=rang(M^*)\Rightarrow SI$

Solución para $a\neq 1$ $a\neq -1$, $M^*=\left(\begin{array}{cc|c}1 & -a & 2\\ a & -1 & a+1\end{array}\right)\approx F_2-aF_1\approx\left(\begin{array}{cc|c}1 & -a & 2\\ 0 & a^2-1 & 1-a\end{array}\right)\Rightarrow y=\frac{1-a}{a^2-1}=\frac{1-a}{(a-1)(a+1)}\Rightarrow$

$\Rightarrow \boxed{y=\frac{-1}{a+1}}$. En la 1ª ecuación : $x-ay=2\Rightarrow x=2+ay=2-\frac{a}{a+1}=\frac{2a+2-a}{a+1}\Rightarrow\boxed{x=\frac{a+2}{a+1}}$

b) Si $y=2\Rightarrow\frac{-1}{a+1}=2\Rightarrow -1=2a+2\Rightarrow 2a=-3\Rightarrow\boxed{a=\frac{-3}{2}}$

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64) a) $A_2=\begin{pmatrix}1 & 1\\ -1 & 9\end{pmatrix}\Rightarrow\boxed{|A_2|=10}$

b) $A_3=\begin{pmatrix}1 & 1 & 1\\ -1 & 9 & 1\\ -1 & -1 & 9\end{pmatrix}\Rightarrow|A_3|=\begin{vmatrix}1 & 1 & 1\\ -1 & 9 & 1\\ -1 & -1 & 9\end{vmatrix}=F_i+F_1=\begin{vmatrix}1 & 1 & 1\\ 0 & 10 & 2\\ 0 & 0 & 10\end{vmatrix}\Rightarrow\boxed{|A_3|=10^2}$

c) $A_4=\begin{pmatrix}1 & 1 & 1 & 1\\ -1 & 9 & 1 & 1\\ -1 & -1 & 9 & 1\\ -1 & -1 & -1 & 9\end{pmatrix}\Rightarrow|A_4|=\begin{vmatrix}1 & 1 & 1 & 1\\ -1 & 9 & 1 & 1\\ -1 & -1 & 9 & 1\\ -1 & -1 & -1 & 9\end{vmatrix}=F_i+F_1=\begin{vmatrix}1 & 1 & 1 & 1\\ 0 & 10 & 2 & 2\\ 0 & 0 & 10 & 2\\ 0 & 0 & 0 & 10\end{vmatrix}\Rightarrow\boxed{|A_4|=10^4}$

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65) Calculamos $|A|=\begin{vmatrix}2 & a+1 & 1\\ 2a & 0 & 1\\ 2 & 0 & a+1\end{vmatrix}=0\Rightarrow -2(a+1)(a^2+a-1)=0\Rightarrow\left\langle\begin{array}{l}a=-1\\ a=\frac{-1\pm\sqrt{5}}{2}\end{array}\right.$

- Caso 1.- Si $a\neq -1$, , $a\neq\frac{-1\pm\sqrt{5}}{2}$ $\Rightarrow|A|\neq 0\Rightarrow rang(A)=3$

- Caso 2.- Si $a=-1$, $rang(A)=rang\begin{pmatrix}2 & 0 & 1\\ -2 & 0 & 1\\ 2 & 0 & 0\end{pmatrix}=2$

- Caso 3.- Si $a=\frac{-1+\sqrt{5}}{2}$, $rang(A)=rang\begin{pmatrix} 2 & 1+\frac{-1+\sqrt{5}}{2} & 1 \\ -1+\sqrt{5} & 0 & 1 \\ 2 & 0 & 1+\frac{-1+\sqrt{5}}{2} \end{pmatrix}=$

$$=rang\begin{pmatrix} 2 & \frac{1+\sqrt{5}}{2} & 1 \\ -1+\sqrt{5} & 0 & 1 \\ 2 & 0 & \frac{1+\sqrt{5}}{2} \end{pmatrix}$$; Como $\begin{vmatrix} (1+\sqrt{5})/2 & 1 \\ 0 & 1 \end{vmatrix}=\frac{1+\sqrt{5}}{2}\neq 0 \Rightarrow rang(A)=2$

- Caso 4.- Si $a=\frac{-1-\sqrt{5}}{2}$, $rang(A)=rang\begin{pmatrix} 2 & 1+\frac{-1-\sqrt{5}}{2} & 1 \\ -1-\sqrt{5} & 0 & 1 \\ 2 & 0 & 1+\frac{-1-\sqrt{5}}{2} \end{pmatrix}=$

$$=rang\begin{pmatrix} 2 & \frac{1-\sqrt{5}}{2} & 1 \\ -1-\sqrt{5} & 0 & 1 \\ 2 & 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$; Como $\begin{vmatrix} (1-\sqrt{5})/2 & 1 \\ 0 & 1 \end{vmatrix}=\frac{1-\sqrt{5}}{2}\neq 0 \Rightarrow rang(A)=2$

b) Si $a=1,\ A=\begin{pmatrix} 2 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix} \Rightarrow \boxed{A^{-1}=\begin{pmatrix} 0 & 1 & -1/2 \\ 1/2 & -1/2 & 0 \\ 0 & -1 & 1 \end{pmatrix}}$

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66) $\begin{cases} x-2y+z-3v=-4 \\ x+2y+z+3v=1 \\ 2x-4y+2z-6v=-8 \\ 2x+2z=0 \end{cases} \Rightarrow$ Matriz ampliada asociada: $rang(M^*)=rang\left(\begin{array}{cccc|c} 1 & -2 & 1 & -3 & -4 \\ 1 & 2 & 1 & 3 & 4 \\ 2 & -4 & 2 & -6 & -8 \\ 2 & 0 & 2 & 0 & 0 \end{array}\right)=$

$$\begin{matrix} F_2-F_1 \\ =F_3-2F_1 \\ F_4-2F_1 \end{matrix}\ =rang\left(\begin{array}{cccc|c} 1 & -2 & 1 & -3 & -4 \\ 0 & 4 & 0 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 6 & 8 \end{array}\right)=F_4-F_2=\left(\begin{array}{cccc|c} 1 & -2 & 1 & -3 & -4 \\ 0 & 4 & 0 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)\Rightarrow rang(M)=2=rang(M^*)\Rightarrow SCI$$

Solución.- El sistema equivalente sería $\begin{Bmatrix} x-2y+z-3v=-4 \\ 4y+6v=8 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x-2y+z-3v=-4 \\ 2y+3v=4 \end{Bmatrix}$ Si $z=\lambda$, $v=\mu$, se tiene:

$2y+3v=4 \Rightarrow y=\frac{4-3v}{2}=\frac{4-3\mu}{2}$.

A su vez: $x-2y+z-3v=-4 \Rightarrow x=-4+2y-z+3v=-4+(4-3\mu)-\lambda+3\mu=-\lambda$

Luego solución: $\boxed{\left\{\left(-\lambda, \frac{4-3\mu}{2}, \lambda, \mu\right) \middle| \lambda, \mu \in R\right\}}$

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67) Sean x = billetes de 50€ , y = billetes de 20€, y z = billetes de 10€

Obtenemos el sistema: $\left\{\begin{array}{c} x+y+z=225 \\ 50x+20y+10z=7000 \\ x+z=2y \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x+y+z=225 \\ 5x+2y+z=700 \\ x-2y+z=0 \end{array}\right\}$

Matriz asociada: $M^*=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 225 \\ 5 & 2 & 1 & 700 \\ 1 & -2 & 1 & 0 \end{array}\right) \approx \begin{array}{c} F_2-5F_1 \\ F_3-F_1 \end{array} \approx \left(\begin{array}{ccc|c} 1 & 2 & -3 & 225 \\ 0 & -3 & -4 & -425 \\ 0 & -3 & 0 & -225 \end{array}\right) \approx F_3-F_2 \approx \left(\begin{array}{ccc|c} 1 & 2 & -3 & 225 \\ 0 & -3 & -4 & -425 \\ 0 & 0 & 4 & 200 \end{array}\right)$

Se tiene $\left\{\begin{array}{c} x+y+z=225 \Rightarrow x=225-y-z=225-75-50=100 \\ -3y-4z=-425 \Rightarrow -3y=-425+4\cdot 50=-225 \Rightarrow y=75 \\ 4z=200 \Rightarrow z=50 \end{array}\right\} \Rightarrow$ $\boxed{x=100}$; $\boxed{y=75}$; $\boxed{z=50}$

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68) $\left\{\begin{array}{cccc} 4x & +4\lambda y & +2z & = & 2\lambda \\ \lambda x & +y & -\lambda z & = & \lambda \\ 4\lambda x & +4\lambda y & +\lambda z & = & 9 \end{array}\right.$; Matrices asociadas: : $M=\begin{pmatrix} 4 & 4\lambda & 2 \\ \lambda & 1 & -1 \\ 4\lambda & 4\lambda & \lambda \end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c} 4 & 4\lambda & 2 & 2\lambda \\ \lambda & 1 & -1 & \lambda \\ 4\lambda & 4\lambda & \lambda & 9 \end{array}\right)$.

$$|M|=\begin{vmatrix} 4 & 4\lambda & 2 \\ \lambda & 1 & -1 \\ 4\lambda & 4\lambda & \lambda \end{vmatrix}=-10\lambda^3+12\lambda^2-2\lambda \Rightarrow -2\lambda(5\lambda^2-6\lambda+1)=0 \Rightarrow \left\langle \begin{array}{l} \lambda=0 \\ 5\lambda^2-6\lambda+1=0 \Rightarrow \left\langle \begin{array}{l} \lambda=1 \\ \lambda=1/5 \end{array}\right. \end{array}\right. .$$

- Caso 1.- Si $\lambda \neq 0, , \lambda \neq 1, \lambda \neq 1/5,\ |M| \neq 0 \Rightarrow rang(M)=3=rang(M^*) \Rightarrow SCD$
- Caso 2.- Si $\lambda=0,\ rang(M^*)=rang\left(\begin{array}{ccc|c} 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{array}\right) \Rightarrow rang(M)=2 \neq 3=rang(M^*) \Rightarrow SI$
- Caso 3.- Si $\lambda=1,\ rang(M^*)=rang\left(\begin{array}{ccc|c} 2 & 2 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 4 & 4 & 1 & 9 \end{array}\right)=F_1 \leftrightarrow F_2=rang\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 2 & 1 & 1 \\ 4 & 4 & 1 & 9 \end{array}\right)=\begin{array}{c} F_2-2F_1 \\ F_3-4F_1 \end{array}=$

$$=rang\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 5 & 5 \end{array}\right)=F_2/(-3)=rang\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 5 & 5 \end{array}\right)=F_3-5F_2=rang\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 10/3 \end{array}\right) \Rightarrow$$

$\Rightarrow rang(M)=2 \neq 3=rang(M^*) \Rightarrow SI$

- Caso 4.- Si $\lambda=\frac{1}{5},\ M^*=\left(\begin{array}{ccc|c} 4 & 4/5 & 2 & 2 \\ 1/5 & 1 & -1/5 & 1/5 \\ 4/5 & 4/5 & 1/5 & 9 \end{array}\right)$.

Como $|M| \neq 0 \Rightarrow rang(M) < 3$, pero $\begin{vmatrix} 4 & 4/5 \\ 1/5 & 1 \end{vmatrix} = 4 - \frac{4}{25} = \frac{96}{25} \neq 0$, luego $rang(M) = 2$

A su vez, tomando el menor de M*: $\begin{vmatrix} 4/5 & 2 & 2 \\ 1 & -1/5 & 1/5 \\ 4/5 & 1/5 & 9 \end{vmatrix} = \frac{-2304}{125} \neq 0 \Rightarrow rang(M^*) = 3$

Luego, $rang(M) = 2 \neq 3 = rang(M^*) \Rightarrow SI$

b) Si $\lambda = -1$, $M^* = \left(\begin{array}{ccc|c} 4 & -4 & 2 & -2 \\ -1 & 1 & 1 & -1 \\ -4 & -4 & -1 & 9 \end{array}\right) \approx F_3 + F_1 \approx \left(\begin{array}{ccc|c} 4 & -4 & 2 & -2 \\ -1 & 1 & 1 & -1 \\ 0 & -8 & 1 & 7 \end{array}\right) \approx F_1 \leftrightarrow F_2 \approx \left(\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 4 & -4 & 2 & -2 \\ 0 & -8 & 1 & 7 \end{array}\right) \approx$

$\approx F_2 + 4F_1 \approx \left(\begin{array}{ccc|c} -1 & 1 & 1 & -1 \\ 0 & 0 & 6 & -6 \\ 0 & -8 & 1 & 7 \end{array}\right) \Rightarrow$ El sistema será equivalente a : $\begin{cases} -x + y + z = -1 \\ 6z = -6 \\ -8y + z = 7 \end{cases}$, resolviendo :

Solución : $\boxed{x = -1}$, $\boxed{y = -1}$, $\boxed{z = -1}$

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69) Sea el sistema $\begin{cases} 2x - y = \lambda \\ \lambda x - 2y = 4 \\ 3x - y = 2 \end{cases}$; Matrices asociadas: $M = \begin{pmatrix} 2 & -1 \\ \lambda & -2 \\ 3 & -1 \end{pmatrix}$ y $M^* = \left(\begin{array}{cc|c} 2 & -1 & \lambda \\ \lambda & -2 & 4 \\ 3 & -1 & 2 \end{array}\right)$

a) Como $\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = 1 \neq 0 \Rightarrow rang(M) = 2$; $|M^*| = \begin{vmatrix} 2 & -1 & \lambda \\ \lambda & -2 & 4 \\ 3 & -1 & 2 \end{vmatrix} = -\lambda^2 + 8\lambda - 12 = 0 \Rightarrow \begin{cases} \lambda = 2 \\ \lambda = 6 \end{cases}$

- <u>Caso 1</u>.- Si $\lambda \neq 2, \lambda \neq 6 \Rightarrow |M^*| \neq 0 \Rightarrow rang(M^*) = 3$, como $rang(M) = 2 \Rightarrow SI$
- <u>Caso 2</u>.- Si $\lambda = 2 \Rightarrow |M^*| = 0 \Rightarrow rang(M^*) < 3$. Como $M^* = \left(\begin{array}{cc|c} 2 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -1 & 2 \end{array}\right)$ y $\begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} = -2 \neq 0 \Rightarrow$

  $rang(M^*) = 2 = rang(M) \Rightarrow SCD$
- <u>Caso 3</u>.- Si $\lambda = 6 \Rightarrow |M^*| = 0 \Rightarrow rang(M^*) < 3$. Como $M^* = \left(\begin{array}{cc|c} 2 & -1 & 6 \\ 6 & -2 & 4 \\ 3 & -1 & 2 \end{array}\right)$ y $\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} = 1 \neq 0 \Rightarrow$

  $rang(M^*) = 2 = rang(M) \Rightarrow SCD$

b) Si $\lambda = 2 \Rightarrow M^* = \left(\begin{array}{cc|c} 2 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -1 & 2 \end{array}\right) \approx F_2 - F_1 \approx \left(\begin{array}{cc|c} 2 & -1 & 2 \\ 0 & -1 & 2 \\ 3 & -1 & 2 \end{array}\right) \approx 2F_3 - 3F_1 \approx \left(\begin{array}{cc|c} 2 & -1 & 2 \\ 0 & -1 & 2 \\ 0 & 1 & -2 \end{array}\right) \approx F_3 + F_2 \approx \left(\begin{array}{cc|c} 2 & -1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$

$\Rightarrow \begin{cases} 2x - y = 2 \\ -y = 2 \end{cases} \Rightarrow \boxed{x = 0}$, $\boxed{y = -2}$

Si $\lambda=6 \Rightarrow M^*=\left(\begin{array}{cc|c} 2 & -1 & 6 \\ 6 & -2 & 4 \\ 3 & -1 & 2 \end{array}\right) \approx \begin{array}{c} F_2-3F_1 \\ 2F_3-3F_1 \end{array} \approx \left(\begin{array}{cc|c} 2 & -1 & 6 \\ 0 & 1 & -14 \\ 0 & 1 & -14 \end{array}\right) \approx F_3-F_2 \approx \left(\begin{array}{cc|c} 2 & -1 & 6 \\ 0 & 1 & -14 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \Rightarrow \left\{\begin{array}{l} 2x-y=6 \\ y=-14 \end{array}\right\} \Rightarrow$

$\Rightarrow \boxed{x=-4}$, $\boxed{y=-14}$

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70) a) Calculamos $|A|=\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}=0 \Rightarrow a^3-3a+2=0 \Rightarrow \left\langle \begin{array}{l} a=1 \\ a=-2 \end{array}\right.$

- Caso 1.- Si $a \neq 1,$, $a \neq -2 \Rightarrow |A| \neq 0 \Rightarrow rang(A)=3$
- Caso 2.- Si $a=1,$, $rang(A)=rang\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}=1$
- Caso 3.- Si $a=-2$, $rang(A)=rang\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$, como $|A|=0 \Rightarrow rang(A)<3$, y $\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}=3 \neq 0$

  $rang(A)=2$

b) Si $a=-1 \Rightarrow A=\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \Rightarrow \boxed{A^{-1}=\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}}$

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71) a) Calculamos $|M|=\begin{vmatrix} m & 1 & 2m \\ m & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}=0 \Rightarrow 2m^2-2m=0 \Rightarrow 2m(m-1)=0 \Rightarrow \boxed{\left\langle \begin{array}{l} m=0 \\ m=1 \end{array}\right.}$

Con lo que existirá M^{-1} $\forall\, m \neq 0, m \neq 1$

b) Para que M^{25} sea invertible $|M^{25}| \neq 0$. Pero $|M^{25}|=|M|^{25} \neq 0$ si $|M| \neq 0$, luego $\forall\, m \neq 0, m \neq 1$

c) Por lo visto en a) para $m=-1$, si existe M^{-1}, operando: $\boxed{M^{-1}=\begin{pmatrix} -1/4 & -3/4 & 1 \\ 1/4 & -1/4 & 1 \\ -1/4 & 1/4 & 0 \end{pmatrix}}$

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72) a) Al ser un sistema homogéneo $rang(M)=rang(M^*)$, luego siempre es compatible. Para que tenga solución distinta de $x=y=z=0$, tiene que ser <u>compatible indeterminado</u>. Luego $rang(M)=rang(M^*)<3$.

Es decir, $rang(M)=\begin{pmatrix} \lambda & 2 & 1 \\ \lambda & -1 & 2 \\ 1 & -\lambda & 2 \end{pmatrix}<3 \Rightarrow |M|=0 \Rightarrow \begin{vmatrix} \lambda & 2 & 1 \\ \lambda & -1 & 2 \\ 1 & -\lambda & 2 \end{vmatrix}=0 \Rightarrow \lambda^2-6\lambda+5=0 \Rightarrow \left\langle \begin{array}{l} \lambda=1 \\ \lambda=5 \end{array}\right.$

b) Si $\lambda=5$, $M=\begin{pmatrix}5&2&1\\5&-1&2\\1&-5&2\end{pmatrix}\approx F_1\leftrightarrow F_3\approx\begin{pmatrix}1&-5&2\\5&-1&2\\5&2&1\end{pmatrix}\approx F_i-5F_1\approx\begin{pmatrix}1&-5&2\\5&-1&2\\5&2&1\end{pmatrix}\approx\begin{pmatrix}1&-5&2\\0&24&-8\\0&27&-9\end{pmatrix}\approx$

$\approx\begin{matrix}F_2/8\\F_3/9\end{matrix}\approx\begin{pmatrix}1&-5&2\\0&3&-1\\0&3&-1\end{pmatrix}\approx F_3-F_2\approx\begin{pmatrix}1&-5&2\\0&3&-1\\0&0&0\end{pmatrix}\Rightarrow\begin{Bmatrix}x-5y+2z=0\\3y-z=0\end{Bmatrix}\Rightarrow$ Haciendo $z=t$

$\begin{Bmatrix}x-5(t/3)+2t=0\Rightarrow x=-t/3\\3y-t=0\Rightarrow y=t/3\end{Bmatrix}\Rightarrow$ Sol: $\boxed{\left\{\left(-\frac{t}{3},\frac{t}{3},t\right)\middle|\ t\in R\right\}}$

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73) Como $AXB=A+B\Rightarrow A^{-1}AXB=A^{-1}(A+B)\Rightarrow XB=A^{-1}(A+B)=A^{-1}A+A^{-1}B=I+A^{-1}B\Rightarrow$

$\Rightarrow XB=I+A^{-1}B\Rightarrow XBB^{-1}=(I+A^{-1}B)B^{-1}\Rightarrow X=IB^{-1}+A^{-1}BB^{-1}=B^{-1}+A^{-1}$

Operando: $X=\begin{pmatrix}-1/2&-1\\-3/2&-2\end{pmatrix}+\begin{pmatrix}1/6&1/3\\-1/6&2/3\end{pmatrix}\Rightarrow\boxed{X=\begin{pmatrix}-1/3&-2/3\\-5/3&-4/3\end{pmatrix}}$

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74) a) Sea el sistema $\begin{cases}x&+ky&-z&=&0\\2x&-y&+2z&=&0\\x&-4y&+kz&=&0\end{cases}$ si ha de tener infinitas soluciones distintas de $x=y=z=0$,

entonces $rang(M)=\begin{pmatrix}1&k&-1\\2&-1&2\\1&-4&k\end{pmatrix}<3\Rightarrow|M|=\begin{vmatrix}1&k&-1\\2&-1&2\\1&-4&k\end{vmatrix}=0\Rightarrow-2k^2+k+15=0\Rightarrow\boxed{\begin{matrix}k=3\\k=-5/2\end{matrix}}$

b) Si $k=3$, por el apartado anterior sabemos que el sistema tendrá infinitas soluciones. Sustituyendo k:

$M=\begin{pmatrix}1&3&-1\\2&-1&2\\1&-4&3\end{pmatrix}\approx\begin{matrix}F_2-2F_1\\F_3-F_1\end{matrix}\approx\begin{pmatrix}1&3&-1\\0&-7&4\\0&-7&4\end{pmatrix}\approx F_3-F_2\approx\begin{pmatrix}1&3&-1\\0&-7&4\\0&0&0\end{pmatrix}\Rightarrow\begin{Bmatrix}x+3y-z=0\\-7y+4z=0\end{Bmatrix}\Rightarrow$ Si $z=\lambda$

$\begin{Bmatrix}x+12\lambda/7-\lambda=0\Rightarrow x=-5\lambda/7\\-7y+4\lambda=0\Rightarrow y=4\lambda/7\end{Bmatrix}\Rightarrow$ Sol: $\boxed{\left\{\left(-\frac{5\lambda}{7},\frac{4\lambda}{7},\lambda\right)\middle|\ \lambda\in R\right\}}$

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75) Sean las matrices: $A=\begin{pmatrix}1&1\\1&-2\end{pmatrix}$, $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

a) $A^2=\begin{pmatrix}1&1\\1&-2\end{pmatrix}\cdot\begin{pmatrix}1&1\\1&-2\end{pmatrix}=\begin{pmatrix}2&-1\\-1&5\end{pmatrix}$.Si $A^2=aA+bI\Rightarrow\begin{pmatrix}2&-1\\-1&5\end{pmatrix}=a\begin{pmatrix}1&1\\1&-2\end{pmatrix}+b\begin{pmatrix}1&0\\0&1\end{pmatrix}\Rightarrow$

$\Rightarrow\begin{pmatrix}2&-1\\-1&5\end{pmatrix}=\begin{pmatrix}a+b&a\\a&-2a+b\end{pmatrix}\Rightarrow\boxed{a=-1}$, $\boxed{b=3}$

b) $A^5=A^2\cdot A^2\cdot A\Rightarrow$ como $A^2=-A+3I$, $A^5=(-A+3I)\cdot(-A+3I)\cdot A=\left(A^2-6A+9I\right)\cdot A=$ sustituyendo

de nuevo el valor de A^2, $=\left(-A+3I-6A+9I\right)\cdot A=\left(-7A+12I\right)\cdot A=-7A^2+12A=$ de nuevo sustituyendo

$$=-7(-A+3I)+12A=7A-21I+12A=19A-21I\Rightarrow \boxed{A^5=\begin{pmatrix}-2 & 19\\ 19 & -59\end{pmatrix}}$$

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76) a) $\begin{cases} x & +ay & -z & = & a \\ ax & & +2z & = & -2 \\ x & & +z & = & -2 \end{cases}$ Matrices asociadas: : $M=\begin{pmatrix}1 & a & -1\\ a & 0 & 2\\ 1 & 0 & 1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c}1 & a & -1 & a\\ a & 0 & 2 & -2\\ 1 & 0 & 1 & -2\end{array}\right)$.

$$|M|=\begin{vmatrix}1 & a & -1\\ a & 0 & 2\\ 1 & 0 & 1\end{vmatrix}=-a(a-2)=0\Rightarrow\left\langle\begin{array}{l}a=0\\ a=2\end{array}\right. .$$

- <u>Caso 1</u>.- Si $a\neq 0,\ ,a\neq 2,\ \ |M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- <u>Caso 2</u>.- Si $a=0,\ rang(M^*)=rang\left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 0 & 2 & -2\\ 1 & 0 & 1 & -2\end{array}\right)=F_3-F_1=rang\left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 0 & 2 & -2\\ 0 & 0 & 2 & -2\end{array}\right)=F_3-F_2=$

$$=rang\left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 0 & 2 & -2\\ 0 & 0 & 0 & 0\end{array}\right)=2=rang(M)\Rightarrow SCI$$

- <u>Caso 3</u>.- Si $a=2,\ rang(M^*)=rang\left(\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 2 & 0 & 2 & -2\\ 1 & 0 & 1 & -2\end{array}\right)=\begin{array}{c}F_2-2F_1\\ F_2-F_1\end{array}=rang\left(\begin{array}{ccc|c}1 & 1 & -1 & 1\\ 2 & 2 & 1 & 1\\ 4 & 4 & 1 & 9\end{array}\right)=\begin{array}{c}F_2/4\\ F_3/2\end{array}=$

$$rang\left(\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & -1 & 1 & -1\\ 0 & -1 & 1 & -2\end{array}\right)=F_3-F_2=rang\left(\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & -1 & 1 & -1\\ 0 & 0 & 0 & -1\end{array}\right)=3\neq 2=rang(M)\Rightarrow SI$$

c) Si $a=0$, el sistema se transformará en $\left(\begin{array}{ccc|c}1 & 0 & -1 & 0\\ 0 & 0 & 2 & -2\\ 0 & 0 & 0 & 0\end{array}\right)\Rightarrow\left\{\begin{array}{l}x-z=0\Rightarrow x=z\\ 2z=-2\Rightarrow z=-1\end{array}\right\}\Rightarrow$ Si $y=\lambda$, se tendrá:

Sol: $\boxed{\{(-1,\lambda,-1)|\ \lambda\in R\}}$

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77) a) $\left|\begin{pmatrix}2 & 4 & 6\\ 6 & 0 & 3\\ \alpha & \beta & \gamma\end{pmatrix}^4\right|=\begin{vmatrix}2 & 4 & 6\\ 6 & 0 & 3\\ \alpha & \beta & \gamma\end{vmatrix}^4=\left[2\cdot\begin{vmatrix}1 & 2 & 3\\ 6 & 0 & 3\\ \alpha & \beta & \gamma\end{vmatrix}\right]^4=2^4\cdot\begin{vmatrix}1 & 2 & 3\\ 6 & 0 & 3\\ \alpha & \beta & \gamma\end{vmatrix}^4=2^4\cdot 3^4=\boxed{1296}$

b) $\begin{vmatrix}10 & 20 & 30\\ 2 & 0 & 1\\ 3\alpha & 3\beta & 3\gamma\end{vmatrix}=10\cdot 3\cdot\begin{vmatrix}1 & 2 & 3\\ 2 & 0 & 1\\ \alpha & \beta & \gamma\end{vmatrix}=10\cdot\begin{vmatrix}1 & 2 & 3\\ 6 & 0 & 3\\ \alpha & \beta & \gamma\end{vmatrix}=10\cdot 3=\boxed{30}$

c) $\begin{vmatrix} 3\alpha+2 & 3\beta+4 & 3\gamma+6 \\ 2\alpha & 2\beta & 2\gamma \\ \alpha+6 & \beta & \gamma+3 \end{vmatrix} = 2\cdot\begin{vmatrix} 3\alpha+2 & 3\beta+4 & 3\gamma+6 \\ \alpha & \beta & \gamma \\ \alpha+6 & \beta & \gamma+3 \end{vmatrix} = F_3 - F_2 = 2\cdot\begin{vmatrix} 3\alpha+2 & 3\beta+4 & 3\gamma+6 \\ \alpha & \beta & \gamma \\ 6 & 0 & 3 \end{vmatrix} = F_1 - 3F_2 =$

$= 2\cdot\begin{vmatrix} 2 & 4 & 6 \\ \alpha & \beta & \gamma \\ 6 & 0 & 3 \end{vmatrix} = 2\cdot 2\cdot\begin{vmatrix} 1 & 2 & 3 \\ \alpha & \beta & \gamma \\ 6 & 0 & 3 \end{vmatrix} = F_2 \leftrightarrow F_3 = -2\cdot 2\cdot\begin{vmatrix} 1 & 2 & 3 \\ 6 & 0 & 3 \\ \alpha & \beta & \gamma \end{vmatrix} = -4\cdot 3 = \boxed{-12}$

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78) a) $\begin{cases} 2x & +my & +3z & = & 3 \\ x & +y & -2z & = & 0 \\ 5x & (m+1)y & +z & = & 9 \end{cases}$ Matrices asociadas: : $M = \begin{pmatrix} 2 & m & 3 \\ 1 & 1 & -2 \\ 5 & m+1 & 1 \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} 2 & m & 3 & 3 \\ 1 & 1 & -2 & 0 \\ 5 & m+1 & 1 & 9 \end{array}\right)$.

$|M| = \begin{vmatrix} 2 & m & 3 \\ 1 & 1 & -2 \\ 5 & m+1 & 1 \end{vmatrix} = -4m - 6 = 0 \Rightarrow m = -\frac{3}{2}$.

- Caso 1.- Si $m \neq -\frac{3}{2}$, $|M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$
- Caso2.- $m = -\frac{3}{2}$, $rang(M^*) = rang\left(\begin{array}{ccc|c} 2 & -3/2 & 3 & 3 \\ 1 & 1 & -2 & 0 \\ 5 & -1/2 & 1 & 9 \end{array}\right) = F_2 \leftrightarrow F_1 = rang\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3/2 & 3 & 3 \\ 5 & -1/2 & 1 & 9 \end{array}\right) = \begin{matrix} F_2 - 2F_1 \\ F_3 - 5F_1 \end{matrix} =$

$= rang\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -7/2 & 7 & 3 \\ 0 & -11/2 & 11 & 9 \end{array}\right) = 11F_2 - \frac{7}{2}F_3 = rang\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -7/2 & 7 & 3 \\ 0 & 0 & 0 & -15 \end{array}\right) = 3 \neq 2 = rang(M) \Rightarrow SI$

b) Si $m = 0$, sustituyendo: $\left(\begin{array}{ccc|c} 2 & 0 & 3 & 3 \\ 1 & 1 & -2 & 0 \\ 5 & 1 & 1 & 9 \end{array}\right) \approx F_1 \leftrightarrow F_2 \approx \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 0 & 3 & 3 \\ 5 & 1 & 1 & 9 \end{array}\right) \approx \begin{matrix} F_2 - 2F_1 \\ F_3 - 5F_1 \end{matrix} \approx \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 7 & 3 \\ 0 & -4 & 11 & 9 \end{array}\right) \approx$

$\approx \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 7 & 3 \\ 0 & -4 & 11 & 9 \end{array}\right) \approx F_3 - 2F_2 \approx \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -2 & 7 & 3 \\ 0 & 0 & -3 & 3 \end{array}\right) \Rightarrow \boxed{z = -1}$; Sustituyendo en la 2ª ecuación:

$-2y - 7 = 3 \Rightarrow -2y = 10 \Rightarrow \boxed{y = -5}$. Por último sustituyendo en la 1ª ecuación: $x - 5 + 2 = 0 \Rightarrow \boxed{x = 3}$

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79) La matriz $A = \begin{pmatrix} 1 & a & 1 \\ 0 & 1 & 0 \\ 0 & 1 & a \end{pmatrix}$ 4tendrá inversa si su determinante es distinto de cero. Es decir:

$$|A| = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & 0 \\ 0 & 1 & a \end{vmatrix} \neq 0 \Rightarrow \boxed{a \neq 0}$$

Suponiendo que $a \neq 0$, calculamos la inversa de la matriz A en función de a utilizando el método de Gauss-Jordan:

$$\left(\begin{array}{ccc|ccc} 1 & a & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & a & 0 & 0 & 1 \end{array}\right) \approx F_3 - F_2 \approx \left(\begin{array}{ccc|ccc} 1 & a & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & -1 & 1 \end{array}\right) \approx F_3 / a \approx \left(\begin{array}{ccc|ccc} 1 & a & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/a & 1/a \end{array}\right) \approx F_1 - F_3 \approx$$

$$\approx \left(\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 1/a & -1/a \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/a & 1/a \end{array}\right) \approx F_1 - aF_2 \approx \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & (1-a^2)/a & -1/a \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1/a & 1/a \end{array}\right) \Rightarrow$$

$$\Rightarrow \boxed{A^{-1} = \begin{pmatrix} 1 & (1-a^2)/a & -1/a \\ 0 & 1 & 0 \\ 0 & -1/a & 1/a \end{pmatrix}}$$

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80) a) $rang(A) = rang \begin{pmatrix} m-1 & 1 & m & 1 \\ 1 & m-1 & m & 1 \\ 1 & 1 & 2 & m-1 \end{pmatrix} = F_1 \leftrightarrow F_3 = rang \begin{pmatrix} 1 & 1 & 2 & m-1 \\ 1 & m-1 & m & 1 \\ m-1 & 1 & m & 1 \end{pmatrix} = \begin{matrix} F_2 - F_1 \\ F_3 - (m-1)F_1 \end{matrix} =$

$$= rang \begin{pmatrix} 1 & 1 & 2 & m-1 \\ 0 & m-2 & m-2 & 2-m \\ 0 & 2-m & -m+2 & 2m-m^2 \end{pmatrix} = F_3 + F_2 = rang \begin{pmatrix} 1 & 1 & 2 & m-1 \\ 0 & m-2 & m-2 & 2-m \\ 0 & 0 & 0 & 2+m-m^2 \end{pmatrix}$$

- Caso 1.- Si $2+m-m^2 \neq 0 \Rightarrow (2-m)(1+m) \neq 0 \Rightarrow m \neq 2, m \neq -1$, $rang(A) = 3$
- Caso 2.- Si $m = 2$, sustituyendo en la última matriz: $rang \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1$
- Caso 3.- Si $m = -1$, sustituyendo en la última matriz $rang \begin{pmatrix} 1 & 1 & 2 & -2 \\ 0 & -3 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2$

b) Si $m = 0$, $A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ Diagonalizando A:

$$\begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & -1 \end{pmatrix} \approx \begin{matrix} F_2 + F_1 \\ F_3 - F_1 \end{matrix} = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{pmatrix} \Rightarrow \text{El sistema quedará: } \begin{Bmatrix} -x+y+t=0 \\ 2t=0 \\ 2y+2z=0 \end{Bmatrix} \Rightarrow t = 0,$$

sustituyendo: $\left.\begin{matrix} -x+y=0 \\ y+z=0 \end{matrix}\right\} \Rightarrow$ Llamando $y=\lambda$, se obtiene: Sol: $\boxed{\{(\lambda,\lambda,-\lambda,0) \mid \lambda \in R\}}$

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81) a) Sea el sistema $\begin{cases} x+2y-z=0 \\ 2x-y+z=3 \end{cases}$. Matrices asociadas: $M=\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$ y $M^*=\begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 3 \end{pmatrix}$

$$rang(M^*)=rang\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 3 \end{array}\right)=F_2-2F_1=rang\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -5 & 3 & 3 \end{array}\right)=2=rang(M)<\text{nª incónitas},\ SCI$$

b) Si añadimos la ecuación: $z=0$, las nuevas matrices del sistema quedarían: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & -1 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right)$

diagonalizando: $F_2-2F_1=rang\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -5 & 3 & 3 \\ 0 & 0 & 1 & 0 \end{array}\right)=3=rang(M)\Rightarrow SCD$

c) Si sumamos las ecuaciones del sistema $\begin{cases} x+2y-z=0 \\ 2x-y+z=3 \end{cases}$ obtenemos: $3x+y=3$, bastaría con añadir la ecuación $3x+y=0$ para que el sistema fuera incompatible.

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82) a) Dada la matriz $A=\begin{pmatrix} -a & 0 & a \\ a & a-1 & 0 \\ 0 & a & a+2 \end{pmatrix}$, calculamos primero $|A|=\begin{vmatrix} -a & 0 & a \\ a & a-1 & 0 \\ 0 & a & a+2 \end{vmatrix}=a\,(2-a)$

- Caso 1.- Si $a\neq 0,\ ,\ a\neq 2\ \Rightarrow |A|\neq 0 \Rightarrow rang(A)=3$
- Caso 2.- Si $a=0,\ rang(A)=rang\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}=2$
- Caso 3.- Si $a=2$, $rang(A)=rang\begin{pmatrix} -2 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 4 \end{pmatrix}=F_2+F_1=rang\begin{pmatrix} -2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}=F_3-2F_2=$

$$rang\begin{pmatrix} -2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}=2$$

b) Existirá A^{-1} si $|A|\neq 0 \Rightarrow \begin{vmatrix} -a & 0 & a \\ a & a-1 & 0 \\ 0 & a & a+2 \end{vmatrix}\neq 0$, por el apartado a) si $\boxed{a\neq 0,2}$

Si $a=1$ procediendo por Gauss-Jordan:

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \approx F_1 \leftrightarrow F_2 \approx \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \approx F_2 + F_1 \approx \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \approx F_2 \leftrightarrow F_3 \approx$$

$$\approx F_2 + F_1 \approx \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \approx F_2 - 3F_3 \approx \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right) \Rightarrow \boxed{A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -3 & -3 & 1 \\ 1 & 1 & 0 \end{pmatrix}}$$

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83) a) Como $|A| = \begin{vmatrix} -a & 0 & a \\ a & a-1 & 0 \\ 0 & a & a+2 \end{vmatrix} = 0 \Rightarrow rang(A) < 3$, luego el sistema nunca será compatible determinado

b) Si $a = 4$ y $B = \begin{pmatrix} 0 \\ -1 \\ b \end{pmatrix}$, las matrices asociadas sistema al serían : $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 4 & 5 & 4 & b \end{array}\right)$. Calculando rangos:

$$rang\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 4 & 5 & 4 & b \end{array}\right) = F_3 - 4F_1 = rang\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 5 & 0 & b \end{array}\right) = rang\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2b+5 \end{array}\right)$$

Se tiene: $rang(A) = 2$ y $rang(A^*) = \begin{cases} 2 & si \quad b = -5/2 \\ 3 & si \quad b \neq -5/2 \end{cases}$. Luego el sistema es incompatible si $\boxed{b \neq -\dfrac{5}{2}}$

c) Si $a = 4$ y $B = \begin{pmatrix} 0 \\ c \\ 10 \end{pmatrix}$, las matrices asociadas sistema al serían : $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & c \\ 4 & 5 & 4 & 10 \end{array}\right)$. Calculando rangos:

$$rang\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & c \\ 4 & 5 & 4 & 10 \end{array}\right) = F_3 - 4F_1 = rang\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & c \\ 0 & 5 & 0 & 10 \end{array}\right) = 2F_3 - 5F_2 = ranrang\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & c \\ 0 & 0 & 0 & 20-5c \end{array}\right)$$

Se tiene: $rang(A) = 2$ y $rang(A^*) = \begin{cases} 2 & si \quad 20-5c = 0 \Rightarrow c = 4 \\ 3 & si \quad 20-5c \neq 0 \Rightarrow c \neq 4 \end{cases}$.

Luego el sistema será compatible indeterminado si $\boxed{c = 4}$

<u>Solución</u>: Tomando $\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{Bmatrix} x + z = 0 \Rightarrow x = -z \\ 2y = 4 \Rightarrow y = 2 \end{Bmatrix}$ Si $z = \lambda$, se obtiene: Sol: $\boxed{\{(-\lambda, 4, \lambda) / \lambda \in R\}}$

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84) a) $\begin{cases} x & +y & +kz & = & k \\ x & +ky & +z & = & k^2 \\ kx & +y & +z & = & 1 \end{cases}$. Matrices asociadas: : $M = \begin{pmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} 1 & 1 & k & k \\ 1 & k & 1 & k^2 \\ k & 1 & 1 & 1 \end{array}\right)$.

$$|M|=\begin{vmatrix}1&1&k\\1&k&1\\k&1&1\end{vmatrix}=(1-k)^2(-k-2)=0\Rightarrow\left\langle\begin{matrix}k=1\\k=-2\end{matrix}\right. .$$

- Caso 1.- Si $k\neq1, , k\neq-2,\ |M|\neq0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- Caso 2.- Si $k\neq1,\ rang(M^*)=\left(\begin{array}{ccc|c}1&1&1&1\\1&1&1&1\\1&1&1&1\end{array}\right)=F_i-F_1=rang\left(\begin{array}{ccc|c}1&1&1&1\\0&0&0&0\\0&0&0&0\end{array}\right)=1=rang(M)<3\Rightarrow SCI$
- Caso 3.- Si $k=-2,\ rang(M^*)=\left(\begin{array}{ccc|c}1&1&-2&-2\\1&-2&1&4\\-2&1&1&1\end{array}\right)=\begin{matrix}F_2-F_1\\F_3+2F_1\end{matrix}=rang\left(\begin{array}{ccc|c}1&1&-2&-2\\0&-3&3&6\\0&3&-3&-3\end{array}\right)=F_3+F_2=$

$$=rang\left(\begin{array}{ccc|c}1&1&-2&-2\\0&-3&3&6\\0&0&0&3\end{array}\right)=3\neq2=rang(M)\Rightarrow SI$$

b) Si $k=0$, sustituyendo: $\left(\begin{array}{ccc|c}1&1&0&0\\1&0&1&0\\0&1&1&1\end{array}\right)\approx F_2-F_1\approx\left(\begin{array}{ccc|c}1&1&0&0\\0&-1&1&0\\0&1&1&1\end{array}\right)\approx F_3+F_1\approx\left(\begin{array}{ccc|c}1&1&0&0\\0&-1&1&0\\0&0&2&1\end{array}\right)$

$\boxed{z=\frac{1}{2}}$; Sustituyendo en la 2ª ecuación: $-y+z=0\Rightarrow\boxed{y=\frac{1}{2}}$.. Por último de la 3ª ecuación:

$x+y=0\Rightarrow\boxed{x=-\frac{1}{2}}$

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85) a) $|A|=\begin{vmatrix}2a&-2&a^2\\-1&a&-1\\2&1&a\end{vmatrix}=4-a^2=0\Rightarrow a=\pm2$

- Caso 1.- Si $a\neq\pm2\Rightarrow|A|\neq0\Rightarrow rang(A)=3$
- Caso 2.- Si $a=-2$, sustituyendo matriz: $rang\begin{pmatrix}-4&-2&4\\-1&-2&-1\\2&1&-2\end{pmatrix}=F_1\leftrightarrow F_2=rang\begin{pmatrix}-1&-2&-1\\-4&-2&4\\2&1&-2\end{pmatrix}=$

$$=\begin{matrix}F_2-4F_1\\F_2+2F_1\end{matrix}=rang\begin{pmatrix}-1&-2&-1\\0&6&8\\0&-3&-4\end{pmatrix}=F_2\leftrightarrow F_3=rang\begin{pmatrix}-1&-2&-1\\0&-3&-4\\0&6&8\end{pmatrix}=F_3+2F_2=rang\begin{pmatrix}-1&-2&-1\\0&-3&-4\\0&0&0\end{pmatrix}=2$$

- Caso 3.- Si $a=2$, sustituyendo matriz: $rang\begin{pmatrix}4&-2&4\\-1&2&-1\\2&1&2\end{pmatrix}=F_1\leftrightarrow F_2=rang\begin{pmatrix}-1&2&-1\\4&-2&4\\2&1&2\end{pmatrix}=$

$$=\begin{matrix}F_2+4F_1\\F_2+2F_1\end{matrix}=rang\begin{pmatrix}-1&2&-1\\0&6&0\\0&5&0\end{pmatrix}=6F_3-5F_2=rang\begin{pmatrix}-1&2&-1\\0&6&0\\0&0&0\end{pmatrix}=2$$

b) Sea $a=2$, el sistema en forma matricial sería: $\begin{pmatrix}4&-2&4\\-1&2&-1\\2&1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\1\\b\end{pmatrix}$. Estudiando rangos:

$$rang(M^*)=\left(\begin{array}{ccc|c}4&-2&4&2\\-1&2&-1&1\\2&1&2&b\end{array}\right)=F_1\leftrightarrow F_2=rang\left(\begin{array}{ccc|c}-1&2&-1&1\\4&-2&4&2\\2&1&2&b\end{array}\right)=\begin{matrix}F_2+4F_1\\F_2+2F_1\end{matrix}=rang\left(\begin{array}{ccc|c}-1&2&-1&1\\0&6&0&6\\0&5&0&b+2\end{array}\right)=$$

$$=F_2/6=rang\left(\begin{array}{ccc|c}-1&2&-1&1\\0&1&0&1\\0&5&0&b+2\end{array}\right)=F_3-5F_2=rang\left(\begin{array}{ccc|c}-1&2&-1&1\\0&1&0&1\\0&0&0&b-3\end{array}\right)$$

- Caso 1.- Si $\boxed{b\neq 3}\Rightarrow rang(M)=2, rang(M^*)=3\Rightarrow\boxed{SI}$
- Caso 2.- Si $\boxed{b=3}$, $\Rightarrow rang(M)=2,=rang(M^*)\Rightarrow\boxed{SCI}$. Calculamos solución:

$\left(\begin{array}{ccc|c}-1&2&-1&1\\0&1&0&1\\0&0&0&0\end{array}\right)\Rightarrow\left\{\begin{matrix}-x+2y-z=1\\y=1\end{matrix}\right\}$ Llamando $z=\lambda$, se obtiene: Sol: $\boxed{\{(1-\lambda,1,\lambda)|\ \lambda\in R\}}$

c) Sea $a=1$, el sistema en forma matricial sería: $\begin{pmatrix}2&-2&1\\-1&1&-1\\2&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$. Por Gauss:

$$=\left(\begin{array}{ccc|c}2&-2&1&-1\\-1&1&-1&2\\2&1&1&2\end{array}\right)=F_1\leftrightarrow F_2=rang\left(\begin{array}{ccc|c}-1&1&-1&2\\2&-2&-1&-1\\2&1&1&2\end{array}\right)=F_i+2F_1=rang\left(\begin{array}{ccc|c}-1&1&-1&2\\0&0&-1&3\\0&3&-1&6\end{array}\right)\Rightarrow$$

$\Rightarrow\left\{\begin{matrix}-x+y-z=2\\-z=3\\3y-z=6\end{matrix}\right\}$, luego $\boxed{z=-3}$. Sustituyendo z en la 2ª ecuación: $\boxed{y=1}$. Por último sustituyendo y, z en la 1ª ecuación $\boxed{x=2}$

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86) a) Matrices asociadas: $M=\begin{pmatrix}0&1&m-1\\0&m-1&1\\m-2&0&0\end{pmatrix}$; y $M^*=\left(\begin{array}{ccc|c}0&1&m-1&m\\0&m-1&1&m\\m-2&0&0&m+2\end{array}\right)$.

$$|M|=\begin{vmatrix}0&1&m-1\\0&m-1&1\\m-2&0&0\end{vmatrix}=(m-2)\cdot\begin{vmatrix}1&m-1\\m-1&1\end{vmatrix}=-m(m-2)^2=0\Rightarrow\left\langle\begin{matrix}m=0\\m=2\end{matrix}\right..$$

- Caso 1.- Si $m\neq 0,$, $m\neq 2$, $|M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$

- <u>Caso 2</u>.- Si $m=0$, $rang(M^*)=\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -2 & 0 & 0 & 2 \end{array}\right) = F_3 \leftrightarrow F_1 = rang\left(\begin{array}{ccc|c} -2 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) = F_3+F_2 =$

$= rang\left(\begin{array}{ccc|c} -2 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) = 2 = rang(M) < 3 \Rightarrow SCI$

- <u>Caso 3</u>.- Si $m=2$, $rang(M^*)=\left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right) = F_2 - F_1 = rang\left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right) = 2$.

  Como $rang(M)=1$, SI

b) Si $m=0$, $M^* \approx \left(\begin{array}{ccc|c} -2 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \left\{\begin{array}{l} -2x=2 \Rightarrow x=-1 \\ -y+z=0 \Rightarrow y=z \end{array}\right\}$ Si $z=\lambda$, se obtiene: Sol: $\boxed{\{(-1,\lambda,\lambda) \mid \lambda \in R\}}$

Si $m=1$, sustituyendo en $M^* \approx \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 3 \end{array}\right) \Rightarrow \left\{\begin{array}{c} y=1 \\ z=1 \\ -x=3 \Rightarrow x=-3 \end{array}\right\}$, luego $\boxed{x=-3}$, $\boxed{y=1}$, $\boxed{z=1}$

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87) $rang(A) = rang\begin{pmatrix} 1 & 3 & -2 \\ -1 & 1 & a \\ 2 & 0 & -a \\ a+2 & 0 & a \end{pmatrix} = \begin{array}{c} F_2+F_1 \\ F_3-2F_1 \\ F_4-(a+2)F_1 \end{array} = rang\begin{pmatrix} 1 & 3 & -2 \\ 0 & 4 & a-2 \\ 0 & -6 & 4-a \\ 0 & -3a-6 & 3a+4 \end{pmatrix} = F_2+F_3 =$

$= rang\begin{pmatrix} 1 & 3 & -2 \\ 0 & -2 & 2 \\ 0 & -6 & 4-a \\ 0 & -3a-6 & 3a+4 \end{pmatrix} = F_2/2 = rang\begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 1 \\ 0 & -6 & 4-a \\ 0 & -3a-6 & 3a+4 \end{pmatrix} = \begin{array}{c} F_3-6F_2 \\ F_4-(3a-6)F_2 \end{array} = rang\begin{pmatrix} 1 & 3 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -2-a \\ 0 & 0 & -2 \end{pmatrix} = 3$

Por tanto $\boxed{rang(A)=3}$ $\forall\, a \in R$

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88) a) $|M| = \begin{vmatrix} sen\,x & \cos x & 0 \\ \cos x & -sen\,x & 0 \\ 0 & 0 & 1 \end{vmatrix} = -sen^2 x - \cos^2 x = \boxed{-1}$

b) $M^2 = \begin{pmatrix} sen\,x & \cos x & 0 \\ \cos x & -sen\,x & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} sen\,x & \cos x & 0 \\ \cos x & -sen\,x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boxed{I}$

c) $M^{25} = M^{12\cdot 2+1} = (M^2)^{12} \cdot M = I^{12} \cdot M = I \cdot M = \boxed{M}$

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89) a) Matrices asociadas: $M=\begin{pmatrix}2 & 4 & 0\\ -k^3 & k^2 & k\\ 1 & k & 0\end{pmatrix}$; y $M^*=\left(\begin{array}{ccc|c}2 & 4 & 0 & 4k\\ -k^3 & k^2 & k & 0\\ 1 & k & 0 & k^2\end{array}\right)$.

$$|M|=\begin{vmatrix}2 & 4 & 0\\ -k^3 & k^2 & k\\ 1 & k & 0\end{vmatrix}=-k\cdot\begin{vmatrix}2 & 4\\ 1 & k\end{vmatrix}=-k(2k-4)=0\Rightarrow\left\langle\begin{matrix}k=0\\ k=2\end{matrix}\right. .$$

- Caso 1.- Si $k\neq 0,\ ,k\neq 2,\ |M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- Caso 2.- Si $k=0,\ rang(M^*)=rang\left(\begin{array}{ccc|c}2 & 4 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\end{array}\right)=2=rang(M)<3\Rightarrow SCI$
- Caso 3.- Si $k=2,\ rang(M^*)=rang\left(\begin{array}{ccc|c}2 & 4 & 0 & 8\\ -8 & 4 & 2 & 0\\ 1 & 2 & 0 & 4\end{array}\right)=\begin{matrix}F_1/2\\ F_2/2\end{matrix}=rang\left(\begin{array}{ccc|c}1 & 2 & 0 & 4\\ -4 & 2 & 1 & 0\\ 1 & 2 & 0 & 4\end{array}\right)=\begin{matrix}F_2+4F_1\\ F_3-F_1\end{matrix}=$

$$=rang\left(\begin{array}{ccc|c}1 & 2 & 0 & 4\\ 0 & 10 & 1 & 16\\ 0 & 0 & 0 & 0\end{array}\right)=2=rang(M)<3\Rightarrow SCI$$

b) Si $k=1$, el sistema es compatible determinado. Sustituyendo:

$$M^*=\left(\begin{array}{ccc|c}2 & 4 & 0 & 4\\ -1 & 1 & 1 & 0\\ 1 & 1 & 0 & 1\end{array}\right)\approx F_2\leftrightarrow F_1\approx\left(\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 2 & 4 & 0 & 4\\ 1 & 1 & 0 & 1\end{array}\right)\approx\begin{matrix}F_2-2F_1\\ F_3+F_1\end{matrix}\approx\left(\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 2 & -2 & 4\\ 0 & 2 & 1 & 1\end{array}\right)\approx F_3-F_2\approx\left(\begin{array}{ccc|c}-1 & 1 & 1 & 0\\ 0 & 2 & -2 & 4\\ 0 & 0 & 3 & -3\end{array}\right)$$

$\Rightarrow\left\{\begin{matrix}-x+y+z=0\\ 2y-2z=4\\ 3z=-3\Rightarrow z=-1\end{matrix}\right\}\Rightarrow\left\{\begin{matrix}-x+y-1=0\\ 2y+2=4\Rightarrow y=1\end{matrix}\right\}\Rightarrow\{-x+1-1=0\Rightarrow x=0\}$. Luego: $\boxed{x=0}$, $\boxed{y=1}$, $\boxed{z=-1}$

c) Si $k=2$, el sistema es compatible indeterminado. Sustituyendo:

$M^*\approx\left(\begin{array}{ccc|c}1 & 2 & 0 & 4\\ 0 & 10 & 1 & 16\\ 0 & 0 & 0 & 0\end{array}\right)\approx\begin{pmatrix}1 & 2 & 0 & 4\\ 0 & 10 & 1 & 16\end{pmatrix}\Rightarrow\left\{\begin{matrix}x+2y=4\\ 10y+z=16\end{matrix}\right\}$ Haciendo $y=\lambda$, Sol: $\boxed{\{(4-2\lambda,\lambda,16-10\lambda)|\ \lambda\in R\}}$

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90) a) $|A|=\begin{vmatrix}k & k & k^2\\ 1 & -1 & k\\ 2k & -2 & 2\end{vmatrix}=-4k+4k^3=0\Rightarrow -4k(1-k^2)=0\Rightarrow\left\langle\begin{matrix}k=0\\ k=-1\\ k=1\end{matrix}\right.$

- Caso 1.- Si $k\neq 0,\pm 1\Rightarrow |A|\neq 0\Rightarrow rang(A)=3$

- Caso 2.- Si $k=0$, sustituyendo matriz: $rang\begin{pmatrix}0&0&0\\1&-1&0\\0&-2&2\end{pmatrix}=2$

- Caso 3.- Si $k=1$, sustituyendo matriz: $rang\begin{pmatrix}1&1&1\\1&-1&1\\2&-2&2\end{pmatrix}=\begin{matrix}F_2-F_1\\F_3-2F_1\end{matrix}=rang\begin{pmatrix}1&1&1\\0&-2&0\\0&-4&0\end{pmatrix}=$

$=F_3-2F_2=rang\begin{pmatrix}1&1&1\\0&-2&0\\0&0&0\end{pmatrix}=2$

- Caso 4.- Si $k=-1$, sustituyendo matriz: $rang\begin{pmatrix}-1&-1&1\\1&-1&-1\\-2&-2&2\end{pmatrix}=\begin{matrix}F_2+F_1\\F_3-2F_1\end{matrix}=rang\begin{pmatrix}-1&-1&1\\0&-2&0\\0&0&0\end{pmatrix}=2$

b) Si $k=2$, el sistema queda: $\begin{pmatrix}2&2&4\\1&-1&2\\4&-2&2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}12\\6\\8\end{pmatrix}$. Como por a) $rang(A)=3$, el sistema será compatible determinado. Sustituyendo:

$$A^*=\left(\begin{array}{ccc|c}2&2&4&12\\1&-1&2&6\\4&-2&2&8\end{array}\right)\approx F_2/2\approx\left(\begin{array}{ccc|c}1&1&1&6\\1&-1&2&6\\4&-2&2&8\end{array}\right)\approx\begin{matrix}F_2-F_1\\F_3-4F_1\end{matrix}\approx\left(\begin{array}{ccc|c}1&1&1&6\\0&-2&0&0\\0&-6&-6&-16\end{array}\right)\approx F_3-3F_2\approx$$

$$\approx\left(\begin{array}{ccc|c}1&1&1&6\\0&-2&0&0\\0&0&-6&-16\end{array}\right)$$. Así el sistema se reduce a: $\left\{\begin{array}{c}x+y+2z=6\\-2y=0\Rightarrow y=0\\-6z=-16\Rightarrow z=16/6=8/3\end{array}\right\}$.

Sustituyendo en la primera ecuación: $x+\frac{16}{3}=6\Rightarrow x=\frac{1}{3}$. Luego: $\boxed{x=\frac{1}{3}}$, $\boxed{y=0}$, $\boxed{z=\frac{8}{3}}$

c) Si $k=1$, el sistema queda: $\begin{pmatrix}1&1&1\\1&-1&1\\2&-2&2\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\3\\3\end{pmatrix}$. Estudiamos rangos:

$$\left(\begin{array}{ccc|c}1&1&1&4\\1&-1&1&3\\2&-2&2&3\end{array}\right)\approx\begin{matrix}F_2-F_1\\F_3-2F_1\end{matrix}\approx\left(\begin{array}{ccc|c}1&1&1&4\\0&-2&0&-1\\0&-4&0&-3\end{array}\right)\approx F_3-2F_2\approx\left(\begin{array}{ccc|c}1&1&1&4\\0&-2&0&-1\\0&0&0&-1\end{array}\right)$$. Luego: $rang(A)=2\neq3=rang(A^*)$

El sistema no tiene solución.

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91) a) Estudiaremos el rango de $B=\begin{pmatrix}4&-1&1&-2\\-2&-3&-7&-8\\3&2-a&3+a&3\end{pmatrix}$ mediante menores

Como $\begin{vmatrix} 4 & -1 \\ -2 & -3 \end{vmatrix} = -14 \neq 0 \Rightarrow rang(B) \geq 2$, $\forall\, a$

Tomando $\begin{vmatrix} 4 & -1 & 1 \\ -2 & -3 & -7 \\ 3 & 2-a & 3+a \end{vmatrix} = 40-40a = 0 \Rightarrow a = 1$ y $\begin{vmatrix} 4 & -1 & -2 \\ -2 & -3 & -8 \\ 3 & 2-a & 3 \end{vmatrix} = 36-36a = 0 \Rightarrow a = 1$

Luego: $\boxed{\left\langle \begin{array}{l} Si\ a \neq 1 \Rightarrow rang(B) = 3 \\ Si\ a = 1 \Rightarrow rang(B) = 2 \end{array} \right.}$

b) Si $a = 0$, $A = \begin{pmatrix} 0 & 1 & 2 \\ -2 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Rightarrow |A| = 4 \neq 0 \Rightarrow$ existe A^{-1}. Con lo que si $AX = B \Rightarrow A^{-1}AX = A^{-1}B \Rightarrow$

$\Rightarrow X = A^{-1}B$. Calculando $A^{-1} = \begin{pmatrix} -1/4 & -1/4 & 1/2 \\ 1/2 & -1/2 & -1 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$. Sustituyendo:

$$X = \begin{pmatrix} -1/4 & -1/4 & 1/2 \\ 1/2 & -1/2 & -1 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 4 & -1 & 1 & -2 \\ -2 & -3 & -7 & -8 \\ 3 & 2 & 3 & 3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}}$$

---

92) $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & y & 1 & 1 \\ 1 & 1 & z & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = F_i - F_4 = \begin{vmatrix} x-1 & 0 & 0 & 0 \\ 0 & y-1 & 0 & 0 \\ 0 & 0 & z-1 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} =$ triangular inferior $= \boxed{(x-1)(y-1)(z-1)}$

---

93) a) Matrices asociadas: $M = \begin{pmatrix} 3 & a & 4 \\ 1 & a+1 & 1 \\ a-1 & -a & -3 \end{pmatrix}$; y $M^* = \left(\begin{array}{ccc|c} 3 & a & 4 & 6 \\ 1 & a+1 & 1 & 3 \\ a-1 & -a & -3 & -3 \end{array}\right)$.

$$|M| = \begin{vmatrix} 3 & a & 4 \\ 1 & a+1 & 1 \\ a-1 & -a & -3 \end{vmatrix} = -3a^2 - 8a - 5 = 0 \Rightarrow \left\langle \begin{array}{l} a = -1 \\ a = -5/3 \end{array} \right. .$$

- Caso 1.- Si $a \neq -1, , a \neq -5/3,\ |M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$
- Caso 2.- Si $a = -1$, $rang(M^*) = rang \left(\begin{array}{ccc|c} 3 & -1 & 4 & 6 \\ 1 & 0 & 1 & 3 \\ -2 & 1 & -3 & -3 \end{array}\right) = F_1 \leftrightarrow F_2 = rang \left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 3 & -1 & 4 & 6 \\ -2 & 1 & -3 & -3 \end{array}\right) =$

$$=\begin{matrix}F_2-3F_1\\F_3+2F_1\end{matrix}=rang\left(\begin{array}{ccc|c}1&0&1&3\\0&-1&1&-3\\0&1&-1&3\end{array}\right)=F_3+F_2=rang\left(\begin{array}{ccc|c}1&0&1&3\\0&-1&1&-3\\0&0&0&0\end{array}\right)=2=rang(M)<3\Rightarrow SCI$$

- Caso 3.- Si $a=-5/3,\ rang(M^*)=rang\left(\begin{array}{ccc|c}3&-5/3&4&6\\1&-2/3&1&3\\-8/3&1&-1&-3\end{array}\right)=3F_i=rang\left(\begin{array}{ccc|c}9&-5&12&18\\3&-2&3&9\\-8&3&-3&-9\end{array}\right)=$

$$=F_1\leftrightarrow F_2=rang\left(\begin{array}{ccc|c}3&-2&3&9\\9&-5&12&18\\-8&3&-2&-9\end{array}\right)=\begin{matrix}F_2-3F_1\\3F_3+8F_1\end{matrix}=rang\left(\begin{array}{ccc|c}3&-2&3&9\\0&1&3&-9\\0&-1&-3&45\end{array}\right)=F_3+F_2=$$

$$=rang\left(\begin{array}{ccc|c}3&-2&3&9\\0&1&3&-9\\0&0&0&36\end{array}\right)=3\neq 2=rang(M)\Rightarrow SI$$

b) Si $a=-1,\quad M^*\approx\left(\begin{array}{ccc|c}1&0&1&3\\0&-1&1&-3\\0&0&0&0\end{array}\right)\Rightarrow\left\{\begin{array}{c}x+z=3\Rightarrow x=3-z\\-y+z=-3\Rightarrow y=z+3\end{array}\right\}$ Si $z=\lambda$, se obtiene:

Sol: $\boxed{\{(3-\lambda,3+\lambda,\lambda)|\ \lambda\in R\}}$

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94) $\det(\vec a,\vec b,\vec d)=-1,\quad \det(\vec a,\vec c,\vec d)=3,\quad \det(\vec b,\vec c,\vec d)=-2$

a) $\det(\vec a,3\vec d,\vec b)=3\cdot\det(\vec a,\vec d,\vec b)=C_2\leftrightarrow C_3=-3\cdot\det(\vec a,\vec b,\vec d)=(-3)(-1)=\boxed{3}$

b) $\det(\vec a-\vec b,\vec c,-\vec d)=\det(\vec a,\vec c,-\vec d)-\det(\vec b,\vec c,-\vec d)=-\det(\vec a,\vec c,\vec d)+\det(\vec b,\vec c,\vec d)=-3+(-2)=\boxed{-5}$

c) $\det(\vec d+3\vec b,2\vec a,\vec b-3\vec a+\vec d)=2\cdot\det(\vec d+3\vec b,\vec a,\vec b-3\vec a+\vec d)=C_3+3C_2=2\cdot\det(\vec d+3\vec b,\vec a,\vec b+\vec d)=C_3-C_1=$

$=2\cdot\det(\vec d+3\vec b,\vec a,-2\vec b)=-4\cdot\det(\vec d+3\vec b,\vec a,\vec b)=-4\cdot\left(\det(\vec d,\vec a,\vec b)+\det(3\vec b,\vec a,\vec b)\right)=$ como el 2º determinante es ccero (dos columnas proporcionales) $=-4\cdot\det(\vec d,\vec a,\vec b)=C_2\leftrightarrow C_1=$

$=4\cdot\det(\vec a,\vec d,\vec b)=C_2\leftrightarrow C_3=-4\cdot\det(\vec a,\vec b,\vec d)=(-4)(-1)=\boxed{4}$

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95) **a)** Matrices asociadas: $M=\begin{pmatrix}1&0&-2\\a&-1&1\\2&0&a\end{pmatrix}$: y $M^*=\left(\begin{array}{ccc|c}1&0&-2&2\\a&-1&1&-8\\2&0&a&4\end{array}\right)$.

$$|M|=\begin{vmatrix}1&0&-2\\a&-1&1\\2&0&a\end{vmatrix}=-a-4=0\Rightarrow a=-4\ .$$

- Caso 1.- Si $a\neq -4,\ |M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$

- Caso 2.- Si $a=-4,\ rang(M^*)=rang\begin{pmatrix} 1 & 0 & -2 & \vline & 2 \\ -4 & -1 & 1 & \vline & -8 \\ 2 & 0 & -4 & \vline & 4 \end{pmatrix} = \begin{matrix} F_2+4F_1 \\ F_3-2F_1 \end{matrix} = rang\begin{pmatrix} 1 & 0 & -2 & \vline & 2 \\ 0 & -1 & -7 & \vline & 0 \\ 0 & 0 & 0 & \vline & 0 \end{pmatrix} = 2 =$

  $= rang(M) < 3 \Rightarrow SCI$

b) Si $a=-5$, por **a)** el sistema será compatible determinado

$$M^* = \begin{pmatrix} 1 & 0 & -2 & \vline & 2 \\ -5 & -1 & 1 & \vline & -8 \\ 2 & 0 & -5 & \vline & 4 \end{pmatrix} \approx \begin{matrix} F_2+5F_1 \\ F_3-2F_1 \end{matrix} \approx \begin{pmatrix} 1 & 0 & -2 & \vline & 2 \\ 0 & -1 & -9 & \vline & 2 \\ 0 & 0 & -1 & \vline & 0 \end{pmatrix} \approx \Rightarrow \begin{Bmatrix} x-2z=2 \\ -y-9z=2 \\ -z=0 \Rightarrow z=0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x=2 \\ -y=2 \end{Bmatrix} \Rightarrow$$

Luego: $\boxed{x=2}$, $\boxed{y=-2}$, $\boxed{z=0}$

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96) **a)** Matrices asociadas: $M=\begin{pmatrix} a & 7 & 5 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{pmatrix}$: y $M^*=\begin{pmatrix} a & 7 & 5 & \vline & 0 \\ 1 & a & 1 & \vline & 3 \\ 0 & 1 & 1 & \vline & -2 \end{pmatrix}$.

$$|M| = \begin{vmatrix} a & 7 & 5 \\ 1 & a & 1 \\ 0 & 1 & 1 \end{vmatrix} = a^2 - a - 2 = 0 \Rightarrow \left\langle \begin{matrix} a=-1 \\ a=2 \end{matrix} \right. .$$

- Caso 1.- Si $a \neq -1, a \neq 2$ $|M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$

- Caso 2.- Si $a=-1,\ rang(M^*)=rang\begin{pmatrix} -1 & 7 & 5 & \vline & 0 \\ 1 & -1 & 1 & \vline & 3 \\ 0 & 1 & 1 & \vline & -2 \end{pmatrix} = F_2+F_1 = rang\begin{pmatrix} -1 & 7 & 5 & \vline & 0 \\ 0 & 6 & 6 & \vline & 3 \\ 0 & 1 & 1 & \vline & -2 \end{pmatrix} = F_2 \leftrightarrow F_1 =$

  $= rang\begin{pmatrix} -1 & 7 & 5 & \vline & 0 \\ 0 & 1 & 1 & \vline & -2 \\ 0 & 6 & 6 & \vline & 3 \end{pmatrix} = F_3-6F_2 = rang\begin{pmatrix} -1 & 7 & 5 & \vline & 0 \\ 0 & 1 & 1 & \vline & -2 \\ 0 & 0 & 0 & \vline & 3 \end{pmatrix} = 3 \neq 2 = rang(M) \Rightarrow SI$

- Caso 3.- Si $a=2,\ rang(M^*)=rang\begin{pmatrix} 2 & 7 & 5 & \vline & 0 \\ 1 & 2 & 1 & \vline & 3 \\ 0 & 1 & 1 & \vline & -2 \end{pmatrix} = F_2 \leftrightarrow F_1 = rang\begin{pmatrix} 1 & 2 & 1 & \vline & 3 \\ 2 & 7 & 5 & \vline & 0 \\ 0 & 1 & 1 & \vline & -2 \end{pmatrix} = F_2-2F_1 =$

  $= rang\begin{pmatrix} 1 & 2 & 1 & \vline & 3 \\ 0 & 3 & 3 & \vline & -6 \\ 0 & 1 & 1 & \vline & -2 \end{pmatrix} = F_2 \leftrightarrow F_1 = rang\begin{pmatrix} 1 & 2 & 1 & \vline & 3 \\ 0 & 1 & 1 & \vline & -2 \\ 0 & 3 & 3 & \vline & -6 \end{pmatrix} = F_3-2F_2 = rang\begin{pmatrix} 1 & 2 & 1 & \vline & 3 \\ 0 & 1 & 1 & \vline & -2 \\ 0 & 0 & 0 & \vline & 0 \end{pmatrix} = 2 = rang(M) \Rightarrow SCI$

b) Si $a=4$, por **a)** el sistema será compatible determinado

$$M^*=\left(\begin{array}{ccc|c}4&7&5&0\\1&4&1&3\\0&1&1&-2\end{array}\right)\approx F_2\leftrightarrow F_1\approx\left(\begin{array}{ccc|c}1&4&1&3\\4&7&5&0\\0&1&1&-2\end{array}\right)\approx F_2-4F_1\approx\left(\begin{array}{ccc|c}1&4&1&3\\0&-9&1&-12\\0&1&1&-2\end{array}\right)\approx F_2\leftrightarrow F_3\approx$$

$$\approx\left(\begin{array}{ccc|c}1&4&1&3\\0&1&1&-2\\0&-9&1&-12\end{array}\right)\approx F_3+9F_2\approx\left(\begin{array}{ccc|c}1&4&1&3\\0&1&1&-2\\0&0&10&-30\end{array}\right)\Rightarrow\left\{\begin{array}{c}x+4y+z=3\\y+z=-2\\10z=-30\Rightarrow z=-3\end{array}\right\}\Rightarrow$$

Resolviendo: $\boxed{x=2}$, $\boxed{y=1}$, $\boxed{z=-3}$

c) Si $a=2$, por **a)** el sistema será compatible indeterminado $M^*\approx\left(\begin{array}{ccc|c}1&2&1&3\\0&1&1&-2\\0&0&0&0\end{array}\right)\Rightarrow\left\{\begin{array}{c}x+2y+z=3\\y+z=-2\Rightarrow\end{array}\right\}$ Si $z=\lambda$,

se obtiene: Sol: $\boxed{\{(\lambda+7,-2-\lambda,\lambda)\mid\lambda\in R\}}$

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97) a) Si $XA=B\Rightarrow XAA^{-1}=BA^{-1}\Rightarrow X=BA^{-1}$. Para que esta ecuación tenga solución debe existir A^{-1}, es decir,

$|A|\neq 0$. Así: $|A|=\begin{vmatrix}1&\lambda&0\\1&1&2\\0&-1&-1\end{vmatrix}\neq 0\Rightarrow\lambda+1\neq 0\Rightarrow\boxed{\lambda\neq -1}$

b) Si $\lambda=4$, $A=\begin{pmatrix}1&4&0\\1&1&2\\0&-1&-1\end{pmatrix}$. Calculando: $A^{-1}=\begin{pmatrix}1/5&4/5&8/5\\1/5&-1/5&-2/5\\-1/5&1/5&-3/5\end{pmatrix}$

Como $X=BA^{-1}=\begin{pmatrix}0&1&1\\1&0&-1\\2&1&0\end{pmatrix}\begin{pmatrix}1/5&4/5&8/5\\1/5&-1/5&-2/5\\-1/5&1/5&-3/5\end{pmatrix}\Rightarrow\boxed{X=\begin{pmatrix}0&0&-1\\2/5&3/5&11/5\\3/5&7/5&14/5\end{pmatrix}}$

c) Si $|A^2B|=|A^2|\cdot|B|=|A|^2\cdot|B|$, como $B=\begin{pmatrix}0&1&1\\1&0&-1\\2&1&0\end{pmatrix}\Rightarrow|B|=-1$, y como por a) $|A|=\lambda+1$, se tendrá:

$\boxed{|A^2B|=-(\lambda+1)^2}$

---

98) a) $|A|=\begin{vmatrix}1&1&a&a\\a&1&1&a\\a&a&1&1\\a&a&a&1\end{vmatrix}=C_1-C_2=\begin{vmatrix}0&1&a&a\\a-1&1&1&a\\0&a&1&1\\0&a&a&1\end{vmatrix}=$ desarrollando por la 1ª columna $=-(a-1)\cdot\begin{vmatrix}1&a&a\\a&1&1\\a&a&1\end{vmatrix}=$

$=C_2-C_3=(1-a)\cdot\begin{vmatrix}1&0&a\\a&0&1\\a&a-1&1\end{vmatrix}=$ desarrollando por la 2ª columna $=-(a-1)(1-a)\cdot\begin{vmatrix}1&a\\a&1\end{vmatrix}=$

$$= (1-a)^2 \cdot (1-a^2) = (1-a)^2(1-a)(1+a) = \boxed{(1-a)^3(1+a)}$$

Rango de A.- Si $a \neq 1, a \neq -1$, $|A| \neq 0 \Rightarrow rang(A) = 4$

Si $a = 1$, $rang(A) = rang\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = F_i - F_1 = rang\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 1$

Si $a = -1$, $rang(A) = rang\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix} = F_i + F_1 = rang\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} = rang\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} = 3$

b) Si $a = 1$, $rang(A) = 1$, como el sistema $AX = 0$ es homogéneo, será compatible indeterminado

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x + y + z + w = 0$. Llamando $x = \lambda, y = \mu, z = \theta \Rightarrow w = -\lambda - \mu - \theta$

Luego, solución: $\boxed{\{(\lambda, \mu, \theta, -\lambda - \mu - \theta) \mid \lambda, \mu, \theta \in R\}}$

c) Si $a = -1$, $rang(A) = 3$, como el sistema $AX = 0$ es homogéneo, será compatible indeterminado

$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x + y - z - w = 0 \\ 2y = 0 \Rightarrow y = 0 \\ -2z = 0 \Rightarrow z = 0 \end{cases}$. Llamando $x = \lambda \Rightarrow \lambda - w = 0 \Rightarrow w = \lambda$

solución: $\boxed{\{(\lambda, 0, 0, \lambda) \mid \lambda \in R\}}$

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99) a) Matrices asociadas: $M = \begin{pmatrix} 2 & \lambda & \lambda \\ 1 & 1 & \lambda - 1 \\ \lambda - 1 & 1 & 1 \end{pmatrix}$; y $M^* = \left(\begin{array}{ccc|c} 2 & \lambda & \lambda & 1-\lambda \\ 1 & 1 & \lambda - 1 & -2\lambda \\ \lambda - 1 & 1 & 1 & \lambda - 1 \end{array}\right)$.

$$|M| = \begin{vmatrix} 2 & \lambda & \lambda \\ 1 & 1 & \lambda - 1 \\ \lambda - 1 & 1 & 1 \end{vmatrix} = -\lambda^2 + \lambda + 2 = 0 \Rightarrow \left\langle \begin{matrix} \lambda = -1 \\ \lambda = 2 \end{matrix} \right. .$$

- Caso 1.- Si $\lambda \neq -1$, $\lambda \neq 2$, $|M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$
- Caso 2.- Si $\lambda = -1$, $rang(M^*) = rang\left(\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 1 & -2 & 2 \\ -2 & 1 & 1 & -2 \end{array}\right) = F_3 + F_1 = rang\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 1 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) = 2 = rang(M)$

$\Rightarrow SCI$

- Caso 3.- Si $\lambda=-2$, $rang(M^*)=rang\left(\begin{array}{ccc|c}2&2&2&-1\\1&1&1&-4\\1&1&1&1\end{array}\right)=F_3-F_2=rang\left(\begin{array}{ccc|c}2&2&2&-1\\1&1&1&-4\\0&0&0&5\end{array}\right)=3$

Como $rang(M)=2\Rightarrow SI$

b) Si $\lambda=1$, por **a)** el sistema será compatible determinado

$$M^*=\left(\begin{array}{ccc|c}2&1&1&0\\1&1&0&-2\\0&1&1&0\end{array}\right)\approx F_2\leftrightarrow F_1\approx\left(\begin{array}{ccc|c}1&1&0&-2\\2&1&1&0\\0&1&1&0\end{array}\right)\approx F_2-2F_1\approx\left(\begin{array}{ccc|c}1&1&0&-2\\0&-1&1&4\\0&1&1&0\end{array}\right)\approx F_3+F_2\approx\left(\begin{array}{ccc|c}1&1&0&-2\\0&-1&1&4\\0&0&2&4\end{array}\right)\approx$$

$\Rightarrow\left\{\begin{array}{c}x+y=-2\\-y+z=4\\2z=4\Rightarrow z=2\end{array}\right\}\Rightarrow\left\{\begin{array}{c}x+y=2\\-y+2=4\end{array}\right\}\Rightarrow$ Resolviendo: $\boxed{x=0}$, $\boxed{y=-2}$, $\boxed{z=2}$

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100) a) Si $A=\begin{pmatrix}\alpha&\beta&\gamma\\\gamma&0&\alpha\\1&\beta&\gamma\end{pmatrix}$ y $B=\begin{pmatrix}1\\0\\1\end{pmatrix}$. Para que $\begin{pmatrix}1\\2\\3\end{pmatrix}$ sea solución de $AX=B$, $\begin{pmatrix}\alpha&\beta&\gamma\\\gamma&0&\alpha\\1&\beta&\gamma\end{pmatrix}\cdot\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$

Operando: $\left\{\begin{array}{lllll}\alpha&+2\beta&+3\gamma&=&1\\\gamma&+3\alpha&&=&0\\1&+2\beta&+3\gamma&=&1\end{array}\right.$. En forma matricial: $\begin{pmatrix}1&2&3\\3&0&1\\0&2&3\end{pmatrix}\cdot\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$rang\left(\begin{array}{ccc|c}1&2&3&1\\3&0&1&0\\0&2&3&0\end{array}\right)=F_2-3F_1=rang\left(\begin{array}{ccc|c}1&2&3&1\\0&-6&-8&-3\\0&2&3&0\end{array}\right)=F_2\leftrightarrow F_3=rang\left(\begin{array}{ccc|c}1&2&3&1\\0&2&3&0\\0&-6&-8&-3\end{array}\right)=F_3+3F_2=$$

$=rang\left(\begin{array}{ccc|c}1&2&3&1\\0&2&3&0\\0&0&1&-3\end{array}\right)$. En ecuaciones: $\left\{\begin{array}{c}\alpha+2\beta+3\gamma=1\\2\beta+3\gamma=0\\\gamma=-3\end{array}\right\}\Rightarrow\boxed{\gamma=-3}$. Sustituyendo en la 2ª ecuación

$2\beta-9=0\Rightarrow\boxed{\beta=\frac{9}{2}}$. Sustituyendo en la 1ª ecuación $\alpha+2\beta+3\gamma=1\Rightarrow\alpha+9-9=1\Rightarrow\boxed{\alpha=1}$

b) Si $\beta=\gamma=1\Rightarrow A=\begin{pmatrix}\alpha&1&1\\1&0&\alpha\\1&1&1\end{pmatrix}$, con lo que el sistema $AX=O$, quedará: $\begin{pmatrix}\alpha&1&1\\1&0&\alpha\\1&1&1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

Para que sea compatible determinado, $|A|=\begin{vmatrix}\alpha&1&1\\1&0&\alpha\\1&1&1\end{vmatrix}\neq0$. Operando: $|A|=\alpha-\alpha^2\neq0\Rightarrow$

$\Rightarrow\boxed{\alpha\neq0,\ \alpha\neq1}$

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c) Si $\alpha=-1, \beta=1, \gamma=0 \Rightarrow A=\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$, con lo que el sistema $AX=B$, $\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Operando: $rang\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) = F_3+F_1 = rang\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \end{array}\right)$

En ecuaciones: $\left\{\begin{array}{c} -x+y=1 \\ -z=0 \\ 2y=2 \end{array}\right\} \Rightarrow \boxed{y=1}$, $\boxed{z=0}$, $\boxed{x=0}$

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101) a) Sea la matriz $A=\begin{pmatrix} -1 & -1 & a \\ -3 & 2 & a \\ 0 & a & -1 \end{pmatrix}$ dicha matriz tendrá inversa si $|A| \neq 0$

Operando: $|A| = \begin{vmatrix} -1 & -1 & a \\ -3 & 2 & a \\ 0 & a & -1 \end{vmatrix} = 2a^2-5 \neq 0 \Rightarrow \boxed{a \neq \pm\sqrt{\frac{5}{2}}}$

b) Si $a=2$, por apartado **a)** la matriz A tendrá inversa. Operando: $\boxed{A^{-1} = \begin{pmatrix} 2 & -1 & 2 \\ 1 & -1/3 & 4/3 \\ 2 & -2/3 & 5/3 \end{pmatrix}}$

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102) Sea **x** = precio cuaderno, **y** = precio rotulador, **z** = precio bolígrafo

a) Se plantea el sistema de ecuaciones $\left\{\begin{array}{c} 5x+2y+3z=22 \\ 2x+y+6z=14 \end{array}\right\}$. Despejando y de la 2ª ecuación:

$y=14-2x-6z$. Sustituyendo en la 1ª ecuación: $5x+2(14-2x-6z)+3z=22 \Rightarrow \boxed{x=9z-6}$

Con lo que $\boxed{y=26-24z}$

b) $8x+3y=8(9z-6)+3(26-24z)=$ $\boxed{30\,€}$

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103) Sean $A=\begin{pmatrix} 1 & a & a \\ 1 & a & 1 \\ a-1 & a & 2 \end{pmatrix}$, $B=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $O=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

a) Calculamos los valores para los que $|A|=0$

$A=\begin{vmatrix} 1 & a & a \\ 1 & a & 1 \\ a-1 & a & 2 \end{vmatrix} = -a^3+3a^2-2a=0 \Rightarrow -a(a-1)(a-2)=0 \Rightarrow$ Si $\boxed{a=0,\ a=1,\ a=2}$ no existe A^{-1}

b) Si $a=-2 \Rightarrow |A| \neq 0$ (ver apartado a)), $\boxed{A^{-1}=\begin{pmatrix} -1/12 & 1/3 & -1/4 \\ -5/24 & -1/6 & -1/8 \\ -1/3 & 1/3 & 0 \end{pmatrix}}$

c) Si $a=1 \Rightarrow$, $A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ y el sistema homogéneo $AX=O$ es compatible indeterminado.

Resolviendo: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \approx F_2-F_1 \approx \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ En ecuaciones: $\left\{\begin{matrix} x+y+z=0 \\ y+2z=0 \end{matrix}\right\} \Rightarrow$

Si $z=\lambda$, de la 2ª ecuación: $y=-2\lambda$. sustituyendo en la 1ª ecuación: $x=\lambda$

La solución será $\boxed{\{(\lambda,-2\lambda,\lambda) |\ \lambda \in R\}}$.

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104) Sea la ecuación matricial: $\begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix} \cdot B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

a) Para que tenga solución debe existir $\begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix}^{-1}$ para poder despejar B, $B=\begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Existirá $\begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix}^{-1}$ si $\begin{vmatrix} a & 2 \\ 3 & 7 \end{vmatrix} \neq 0 \Rightarrow 7a-6 \neq 0 \Rightarrow \boxed{a \neq \frac{6}{7}}$

b) Si $a=1 \Rightarrow \exists$ la inversa de $\begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix}$ (ver apartado a).

Sustituyendo y operando $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$, luego $B=\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 5 & 5 \\ -2 & -2 \end{pmatrix}}$

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105) Sea la matriz $A=\begin{pmatrix} 2 & -1 & -3 & 5 \\ 2 & 2 & -1 & a \\ 1 & 1 & 1 & 6 \\ 3 & 1 & -4 & a \end{pmatrix}$

Calculamos: $|A|=\begin{vmatrix} 2 & -1 & -3 & 5 \\ 2 & 2 & -1 & a \\ 1 & 1 & 1 & 6 \\ 3 & 1 & -4 & a \end{vmatrix} = C_1 \leftrightarrow C_2 = -\begin{vmatrix} -1 & 2 & -3 & 5 \\ 2 & 2 & -1 & a \\ 1 & 1 & 1 & 6 \\ 1 & 3 & -4 & a \end{vmatrix} = \begin{matrix} F_2+2F_1 \\ F_3+F_1 \\ F_4+F_1 \end{matrix} = -\begin{vmatrix} -1 & 2 & -3 & 5 \\ 0 & 6 & -7 & a+10 \\ 0 & 3 & -2 & 11 \\ 0 & 5 & -7 & a+5 \end{vmatrix} =$

= desarrollando por la 1ª columna $= -(-1)\cdot\begin{vmatrix} 6 & -7 & a+10 \\ 3 & -2 & 11 \\ 5 & -7 & a+5 \end{vmatrix} = \begin{vmatrix} 6 & -7 & a+10 \\ 3 & -2 & 11 \\ 5 & -7 & a+5 \end{vmatrix} = 12-2a$

Igualando a cero $|A| = 0 \Rightarrow 12-2a = 0 \Rightarrow a = 6$

- Caso 1.- Si $\boxed{a \neq 6 \Rightarrow |A| \neq 0 \Rightarrow rang(A) = 4}$
- Caso 2.- Si $a = 6 \Rightarrow rang(A) < 4$. Sustituyendo: $A = \begin{pmatrix} 2 & -1 & -3 & 5 \\ 2 & 2 & -1 & 6 \\ 1 & 1 & 1 & 6 \\ 3 & 1 & -4 & 6 \end{pmatrix}$

  Tomando el menor: $\begin{vmatrix} 2 & -1 & -3 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 9 \neq 0 \Rightarrow \boxed{a = 6 \Rightarrow rang(A) = 3}$

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106) a) Matrices asociadas : $M = \begin{pmatrix} 4 & 3 & m-1 \\ 1 & -2 & m \\ 5 & m & 1 \end{pmatrix}$ y $M^* = \left(\begin{array}{ccc|c} 4 & 3 & m-1 & 0 \\ 1 & -2 & m & 1 \\ 5 & m & 1 & 1 \end{array}\right)$.

$|M| = \begin{vmatrix} 4 & 3 & m-1 \\ 1 & -2 & m \\ 5 & m & 1 \end{vmatrix} = -m^2+8m-7 = 0 \Rightarrow \left\langle \begin{matrix} m=1 \\ m=7 \end{matrix} \right.$.

- Caso 1.- Si $m \neq 1,\ m \neq 7,\ |M| \neq 0 \Rightarrow rang(M) = 3 = rang(M^*) \Rightarrow SCD$
- Caso 2.- Si $m = 1,\ rang(M^*) = rang\left(\begin{array}{ccc|c} 4 & 3 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{array}\right) = F_1 \leftrightarrow F_2 = rang\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 4 & 3 & 0 & 0 \\ 5 & 1 & 1 & 1 \end{array}\right) = \begin{matrix} F_2-4F_1 \\ F_3-5F_1 \end{matrix} =$

  $= rang\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 11 & -4 & -4 \\ 0 & 11 & -4 & -4 \end{array}\right) = F_3-F_2 = rang\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 11 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right) = 2 = rang(M) \Rightarrow SCI$

- Caso 3.- Si $m = 7,\ rang(M^*) = rang\left(\begin{array}{ccc|c} 4 & 3 & 6 & 0 \\ 1 & -2 & 7 & 0 \\ 5 & 7 & 1 & 1 \end{array}\right) = F_1 \leftrightarrow F_2 = rang\left(\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 4 & 3 & 6 & 0 \\ 5 & 7 & 1 & 1 \end{array}\right) = \begin{matrix} F_2-4F_1 \\ F_3-5F_1 \end{matrix} =$

  $= rang\left(\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 11 & -22 & 0 \\ 0 & 17 & -34 & 1 \end{array}\right) = \begin{matrix} F_2/11 \\ F_3/17 \end{matrix} = rang\left(\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) = F_3-F_2 = rang\left(\begin{array}{ccc|c} 1 & -2 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = 3$

  Como $ang(M) = 2 \Rightarrow SI$

b) Si $m=1$, el sistema es compatible indeterminado (ver apartado a)). Sustituyendo:

$$\left(\begin{array}{ccc|c} 4 & 3 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 5 & 1 & 1 & 1 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 11 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$. En ecuaciones: $\Rightarrow \left\{\begin{array}{l} x-2y+z=1 \\ 11y-4z=-4 \\ 2z=4 \Rightarrow z=2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x-2y+z=1 \\ 11y-4z=-4 \end{array}\right\} \Rightarrow$

Si $z=\lambda$, de la 2ª ecuación: $11y-4\lambda=-4 \Rightarrow y=\frac{4\lambda-4}{11}$. Sustituyendo en la 1ª ecuación:

$x-2\cdot\left(\frac{4\lambda-4}{11}\right)+\lambda=1 \Rightarrow x=\frac{3-3\lambda}{11}$, luego la solución será $\boxed{\left\{\left(\frac{3-3\lambda}{11},\frac{4\lambda-4}{11},\lambda\right) \mid \lambda \in R\right\}}$.

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107) Sean las matrices $A=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

a) $A^2=A\cdot A=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=I$.

Análogamente: $A^3=A^2\cdot A=I\cdot\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}=A$.

Luego, en general:

- $A^n=I$, si n es par
- $A^n=A$, si n es impar

Con lo que $\boxed{A^{15}=A}$ y $\boxed{A^{20}=I}$

b) Si $6X=B-3AX \Rightarrow 6X+3AX=B \Rightarrow (6I+3A)X=B \Rightarrow (6I+3A)^{-1}(6I+3A)X=(6I+3A)^{-1}B \Rightarrow$

$\Rightarrow X=(6I+3A)^{-1}B$. Como $\Rightarrow 6I+3A=\begin{pmatrix} 6 & 0 & 3 \\ 0 & 9 & 0 \\ 3 & 0 & 6 \end{pmatrix} \Rightarrow (6I+3A)^{-1}=\begin{pmatrix} 2/9 & 0 & -1/9 \\ 0 & 1/9 & 0 \\ -1/9 & 0 & 2/9 \end{pmatrix} \Rightarrow$

$$\Rightarrow X=\begin{pmatrix} 2/9 & 0 & -1/9 \\ 0 & 1/9 & 0 \\ -1/9 & 0 & 2/9 \end{pmatrix}\cdot\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}=\boxed{\begin{pmatrix} 2/3 & 0 & -1/3 \\ 0 & 1/3 & 0 \\ -1/3 & 0 & 2/3 \end{pmatrix}}$$

De otra manera.- $6X=B-3AX \Rightarrow 6X+3AX=B \Rightarrow (6I+3A)X=B \Rightarrow 3\cdot(2I+A)X=B$, como $B=3I$

$3\cdot(2I+A)X=3I \Rightarrow (2I+A)X=I \Rightarrow X=(2I+A)^{-1}$

$$\Rightarrow 2I+A=\begin{pmatrix}2&0&1\\0&3&0\\1&0&2\end{pmatrix}\Rightarrow X=(2I+A)^{-1}=\boxed{\begin{pmatrix}2/3&0&-1/3\\0&1/3&0\\-1/3&0&2/3\end{pmatrix}}$$

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108) a) $|A|=\begin{vmatrix}1&2&3\\0&t&2\\3&-1&t\end{vmatrix}=t^2-9t+14=0\Rightarrow\Rightarrow\left\langle\begin{matrix}t=2\\t=7\end{matrix}\right.$

- Caso 1.- Si $t\neq 2,7\Rightarrow|A|\neq 0\Rightarrow rang(A)=3$
- Caso 2.- Si $t=2$, sustituyendo matriz: $rang\begin{pmatrix}1&2&3\\0&2&2\\3&-1&2\end{pmatrix}=F_3-3F_1=rang\begin{pmatrix}1&2&3\\0&2&2\\0&-7&-7\end{pmatrix}=$

$=F_2/2=rang\begin{pmatrix}1&2&3\\0&1&1\\0&-7&-7\end{pmatrix}=F_3+7F_2=rang\begin{pmatrix}1&2&3\\0&1&1\\0&0&0\end{pmatrix}=2$

- Caso 3.- Si $t=7$, sustituyendo matriz: $rang\begin{pmatrix}1&2&3\\0&7&2\\3&-1&7\end{pmatrix}=F_3-3F_1=rang\begin{pmatrix}1&2&3\\0&7&2\\0&-7&-2\end{pmatrix}=$

$=F_3+F_2=rang\begin{pmatrix}1&2&3\\0&7&2\\0&0&0\end{pmatrix}=2$

b) $|A-tI|=\begin{vmatrix}1-t&2&3\\0&0&2\\3&-1&0\end{vmatrix}=14-2t=0\Rightarrow\boxed{t=7}$

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109) a) Matrices asociadas : $M=\begin{pmatrix}-m&m&1\\1&-m&3\\1&-2&-1\end{pmatrix}$ y $M^*=\left(\begin{array}{ccc|c}-m&m&1&0\\1&-m&3&4\\1&-2&-1&0\end{array}\right)$.

$|M|=\begin{vmatrix}-m&m&1\\1&-m&3\\1&-2&-1\end{vmatrix}=-m^2+3m-2=0\Rightarrow\left\langle\begin{matrix}m=1\\m=2\end{matrix}\right.$.

- Caso 1.- Si $m\neq 1,\ m\neq 2,\ |M|\neq 0\Rightarrow rang(M)=3=rang(M^*)\Rightarrow SCD$
- Caso 2.- Si $m=1,\ rang(M^*)=rang\left(\begin{array}{ccc|c}-1&1&1&0\\1&-1&3&4\\2&-2&-1&0\end{array}\right)=\begin{matrix}F_2\leftrightarrow F_1\\F_3+2F_1\end{matrix}=rang\left(\begin{array}{ccc|c}-1&1&1&0\\0&0&4&4\\0&0&1&0\end{array}\right)=F_2/4=$

$$= rang\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = F_3 - F_2 = rang\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 \end{pmatrix} = 3 \neq 2 = rang(M) \Rightarrow SI$$

- Caso 3.- Si $m=2,\ rang(M^*) = rang\begin{pmatrix} -2 & 2 & 1 & 0 \\ 1 & -2 & 3 & 4 \\ 2 & -2 & -1 & 0 \end{pmatrix} = F_1 \leftrightarrow F_2 = rang\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 2 & 1 & 0 \\ 2 & -2 & -1 & 0 \end{pmatrix} = \begin{matrix} F_2 + 2F_1 \\ F_3 - 2F_1 \end{matrix} =$

$$= rang\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & -2 & 7 & 8 \\ 0 & 2 & -7 & -8 \end{pmatrix} = F_3 + F_2 = rang\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & -2 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 2 = rang(M) \Rightarrow SCI$$

b) Si $m=0$**,** el sistema es compatible determinado (ver apartado a)). Sustituyendo:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 4 \\ 2 & -2 & -1 & 0 \end{pmatrix} \approx F_3 - 2F_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 4 \\ 0 & -2 & -7 & -8 \end{pmatrix}$$. En ecuaciones: $\left\{ \begin{matrix} z=0 \\ x+3z=4 \\ -2y-z=-8 \end{matrix} \right\} \Rightarrow \boxed{x=4}$, $\boxed{y=4}$, $\boxed{z=0}$

c) Si $m=2$**,** el sistema es compatible indeterminado (ver apartado a)). Sustituyendo:

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 4 \\ 2 & -2 & -1 & 0 \end{pmatrix} \approx \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & -2 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$. En ecuaciones: $\left\{ \begin{matrix} x-2y+3z=4 \\ -2y+7z=8 \end{matrix} \right\}$. Si $z=\lambda$, de la 2ª ecuación:

$-2y+7\lambda=8 \Rightarrow y=\dfrac{7\lambda-8}{2}$. Sustituyendo en la 1ª ecuación: $x-2\cdot\left(\dfrac{7\lambda-8}{2}\right)+3\lambda=4 \Rightarrow$,

$\Rightarrow x-7\lambda+8+3\lambda=4 \Rightarrow x=4\lambda-4$ luego la solución será $\boxed{\left\{\left(4\lambda-4, \dfrac{7\lambda-8}{2}, \lambda\right) \middle| \lambda \in R\right\}}$.

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110) a) $\begin{vmatrix} 2a-2b & c & 5b \\ 2d-2e & f & 5e \\ -2 & 3 & 10 \end{vmatrix} = 5\cdot\begin{vmatrix} 2a-2b & c & b \\ 2d-2e & f & e \\ -2 & 3 & 2 \end{vmatrix} = C_1 + 2C_5 = 5\cdot\begin{vmatrix} 2a & c & b \\ 2d & f & e \\ 2 & 3 & 2 \end{vmatrix} = 5\cdot 2\cdot\begin{vmatrix} a & c & b \\ d & f & e \\ 1 & 3 & 2 \end{vmatrix} = C_2 \leftrightarrow C_3 =$

$= -5\cdot 2\cdot\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 2 & 3 \end{vmatrix}$. Como $\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 2 & 3 \end{vmatrix} = 3,\ \begin{vmatrix} 2a-2b & c & 5b \\ 2d-2e & f & 5e \\ -2 & 3 & 10 \end{vmatrix} = \boxed{-30}$

b) $\begin{vmatrix} a-1 & b-2 & 2c-6 \\ 2 & 4 & 12 \\ d & e & 2f \end{vmatrix} = \begin{vmatrix} a-1 & b-2 & 2(c-3) \\ 2 & 4 & 2\cdot 6 \\ d & e & 2f \end{vmatrix} = 2\cdot\begin{vmatrix} a-1 & b-2 & c-3 \\ 2 & 4 & 6 \\ d & e & f \end{vmatrix} = 2\cdot\begin{vmatrix} a-1 & b-2 & c-3 \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 \\ d & e & f \end{vmatrix} =$

$= 2\cdot 2\cdot\begin{vmatrix} a-1 & b-2 & c-3 \\ 1 & 2 & 3 \\ d & e & f \end{vmatrix} = F_1 + F_2 == 4\cdot\begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ d & e & f \end{vmatrix} = F_2 \leftrightarrow F_3 = -4\cdot\begin{vmatrix} a & b & c \\ d & e & f \\ 1 & 2 & 3 \end{vmatrix} = (-4)\cdot 3 = \boxed{-12}$

111) Si $AB=BA \Rightarrow \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3a+c & 3b+d \\ a & b \end{pmatrix} = \begin{pmatrix} 3a+b & a \\ 3c+d & c \end{pmatrix} \Rightarrow$ en ecuaciones

$\left\{ \begin{array}{c} 3a+c=3a+b \\ 3b+d=a \\ a=3c+d \\ b=c \end{array} \right\}$. La 1ª y 4ª ecuación son equivalentes, luego $\boxed{B=\begin{pmatrix} 3b+d & b \\ b & d \end{pmatrix}}$
